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Discovering New Theorems: A Study of Extensions of the BCSK Logic*

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1. Theorem Discovery Versus Theorem Proving

Where do new theorems come from? For a tenable answer, some come from a person making a conjecture, some from an open question being posed, some come from an educated guess about what is true. In other words, from what I can tell from various discussions with colleagues, virtually all new theorems come from the mind of some individual. When a purported theorem is brought to my attention with a request to attempt to prove it, I (as is known to so many) invoke the assistance of W. McCune's automated reasoning program OTTER. In most cases, I am not the source of new theorems. An exception is discussed in my notebook titled Proof Shortening. There I show how methodologies formulated to find shorter proofs were in fact applied to find so-called first proofs, which led OTTER to the discovery of three new single axioms for the *BCI* logic.

In this notebook, I focus of theorem discovery of a serendipitous nature, in contrast to the explicit attempt to seek a proof of a theorem offered by some researcher. The area of concern is the *BCSK* logic, whose axioms I give shortly. (Later in this notebook, I focus on two different extensions of *BCSK*, namely, *BCSK+* and *SBPC*.) In another notebook also focusing on this area of logic, I promised to tell one or more stories, stories featuring the discovery of new theorems, new to me, and new to M. Spinks who was the wellspring for my entrance into the *BCSK* logic and its extensions.

The *BCSK* logic can be axiomatized with the following nine axioms (expressed as clauses), where *i* and *j* respectively denote *strong* and *weak* implication.

```
P(i(x,i(y,x))).                      % (A1)
P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).  % (A2)
P(i(i(i(x,y),x),x)).                 % (A3)
P(i(x,j(y,x))).                      % (A4)
P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).  % (A5)
P(i(j(x,j(y,z)),j(y,j(x,z)))).       % (A6)
P(i(j(j(x,y),x),x)).                 % (A7)
P(i(j(i(x,y),y),j(i(y,x),x))).       % (A8)
P(j(i(x,y),j(x,y))).                 % (A9)
```

Variables such as *x*, *y*, and *z* are implicitly universally quantified, meaning "for all". The predicate *P* can be interpreted as "is provable". A subset of these axioms bears an interesting relationship to intuitionistic logic: namely, the first three provide an axiomatization of the implicational fragment of intuitionistic logic.

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The inference rules used in the studies reported here are, respectively, for *condensed detachment* for the function *i* and the function *j*, the following, where " | " denotes logical **or** and "-" denotes logical **not**.

6 [] -P(i(x,y)) | -P(x) | P(y).
7 [] -P(j(x,y)) | -P(x) | P(y).

Hyperresolution, together with the cited clauses, enables an automated reasoning program to apply condensed detachment.

2. The Impetus for My Research

The story begins with a phone conversation between Spinks and me. Spinks introduced me to the *BCSK* logic with a request that I find shorter proofs than he had in hand for two theorems, one of which was a conjunction. R. Veroff and Spinks had successfully collaborated on various studies. Using his sketches methodology, Veroff had found proofs of the three theorems (theses) brought to me by Spinks, the following given in negated form (as targets). The first theorem to prove is Thesis_1; the second to prove is the conjunction (join) of Thesis_2 and Thesis_3.

19 [] -P(i(i(A,B),j(A,B))) | $ANS(THESIS_1).
-P(j(i(A,B),i(j(B,C),j(A,C)))) | $ANS(THESIS_2).
-P(j(i(B,C),i(j(A,B),j(A,C)))) | $ANS(THESIS_3).

Immediately, with reliance on Veroff's three proofs (if memory serves), I began seeking proofs shorter than those that were known at the time. Of the aspects of the approach to proof refinement with respect to length, two were prominent. First, I used ancestor subsumption, which causes the program to compare pairs of paths to the same conclusion, preferring the shorter derivation. Second, I used demodulation to block steps of a proof, one at a time, to prevent their participation. (Demodulation is typically used for simplification and canonicalization.) By blocking the use of some given step, the program is forced to seek other paths to a proof—and it often is a most effective move to make.

I am almost certain that I did not use at the time a 38-step proof that Spinks had sent me by e-mail, one that relies on axioms 1, 2, 4, 5, 6, 7, and 9 and does not rely on the 3-literal clause for (condensed detachment) focusing on the function *i*. On the other hand, I did (I am fairly sure) use two collections of *resonators* sent to me by e-mail from Spinks. A resonator is a formula or an equation that does not itself take on the value **true** or **false** but is instead used to direct a program's reasoning. Its functional pattern is the key, where all variables within a resonator are treated as indistinguishable from each other, just denoting that a variable occurs in the corresponding position. The value assigned to a resonator reflects its conjectured importance: the smaller the value, the higher the priority for being chosen to initiate inference-rule application. A deduced conclusion that matches a resonator, with all variables treated as indistinguishable, is assigned the value assigned to the matching resonator.

My goal was to find a proof of each of the three theses. When and if I had a proof in hand, even if not proofs for all three, I would then begin a study aimed at obtaining short proofs, using the approaches already described.

3. Discovering a New Theorem

Of course, I found something, or I would not be telling this story. Specifically, I found a proof of Thesis 1. For that proof, I did not use ancestor subsumption; that process can slow OTTER very much. After all, I simply was after some proof to consider. I did not directly, as I recall, rely on any of the three Veroff proofs.

Ancestor subsumption and demodulation blocking now came into play, and progress was made. Shorter and still shorter proofs were being found, one of them of especial interest. In particular, that proof did not among its hypotheses *A*3. In other words, only eight of the nine axioms for *BCSK* were used, at the axiomatic or input level, in drawing conclusions.

My next move was to place a percent sign in column 1 of the line occupied by *A*3, which instructs OTTER to treat that line as a comment. I continued my pursuit, with some success, of finding a shorter proof of Thesis 1. I intended to turn my attention to Theses 2 and 3 later. Eventually, I had various short proofs, one of which exhibited a remarkable property.

Indeed, that wonderful proof contained, among its so deduced steps, *A*3. OTTER had proved that *A*3 is in fact dependent on the other eight original axioms; see Section 6 for a short proof in this regard. A most pleasing new theorem had been discovered. I notified Spinks of this dependency and was gratified by his utter surprise. For me, a nice contrast existed. Specifically, the new theorem stating that *A*3 is dependent was essentially discovered, discovered by the program, which contrasts, say, with Thesis 1, a theorem taken from the mind of a researcher. Perhaps, just perhaps, a proof existed with *A*3 unneeded in any way, even at the deduced level, and my next small expedition commenced. Of course, perhaps *A*3 would prove to be required for any proof of Thesis 1, required at the axiom level or at the deduced level.

The obvious approach to take was that which I often rely upon when the goal is to avoid retention of some item designated as unwanted. Therefore, I turned to demodulation, including a demodulator to block the retention of *A*3 when and if deduced. The following illustrates how this is done.

```
list(demodulators).
(P(i(i(i(x, y), x), x)) = junk).  %  A3
(i(x,junk) = junk).
(i(junk,x) = junk).
(j(x,junk) = junk).
(j(junk,x) = junk).
(P(junk) = $T).
end_of_list.
```

Weighting could have been used. With demodulation, items subsumed by a demodulator are also treated as the demodulator itself is treated; with weighting, items similar in functional shape are treated as the corresponding input weight template dictates. It worked: OTTER found an appropriate proof. Later in this notebook, I shall include a nice proof showing *A*3 dependent. So, at this point, I had reduced the focus of the study from nine axioms to eight.

Naturally and directly, the cited success caused me to wonder about other possible dependencies among the original nine axioms. If such existed, then fewer than eight axioms would be needed to axiomatize *BCSK*, and so I come to the next story. However, I cannot tell you the story I intended telling because I am unable to locate the crucial files from five years ago; my file space currently exceeds 65 gigabytes. I am forced, therefore, to tell you the current story (in May 2009), a story that in a way is better than the one I was going to tell you. Indeed, the newer story (which I just enjoyed experiencing) shows that a method I employed those five years ago, and that I just employed again, is powerful and, for some in the future, may prove most useful.

I chose for a second possible dependency *A*6 because of its relation to *A*3. In particular, *A*3 is the third of the given axioms concerned exclusively with the function *i*, and *A*6 is the third of those concerned almost exclusively with the function *j*. Again, my approach was to comment out *A*6 in the input, and I found appropriate proofs. I thus knew that *A*6 was not needed, at the axiomatic level, to find proofs of the three given theses. So, in the context of Thesis 1, I had a proof in which neither *A*3 nor *A*6 was used as an axiom. In that proof (of Thesis 1) *A*3 was not even present as a deduced step, but, perhaps a bit disappointing, *A*6 was among the deduced steps.

Well, emulation was in order. Surely demodulation would take care of the problem. Failure resulted. Further, various attempts at completing a proof in which *A*6 was totally absent did not produce what I was after. I had no choice; indeed, I must make a radical move.

I was thus forced to depart from my usual practice, that of paying little or no attention (in the vast majority of my studies) to the actual proofs themselves. More precisely, my approach typically does not call for a close examination of a completed proof, in detail or as a whole. Instead, I rely on years of experimentation that have given me a feel for which options and which values, if assigned to parameters, are likely to enable the program to complete a given assignment. In other words, I have found that the reading of a proof usually sheds little or no light on how one might proceed to refine it. Instead, such a reading, at least for me, plays a role in the formulation of new strategies and new methodologies.

At this point, the original story and the current story are still somewhat similar, but now they begin to diverge except for relying on the same approach, the powerful method I cited earlier and that I will now offer. In the story I had expected to tell, a particular proof of Thesis 1 would take center stage, and I would extract from it various items. As noted, I could not find that proof. However, after extensive wandering among my files, I found a proof that might serve well, a proof of Thesis 3. In that proof, *A*3 was absent (both as an axiom and as a deduced step), *A*6 was absent as an axiom, and *A*6 was present as a deduced step. The seeds of the (new) method of interest were planted. Indeed, in the following proof, you will find *A*6 as a deduced step, the fifty-seventh. You will also find that this formula is the parent of precisely one step, the fifty-eighth, in the 61-step proof. In the story I would have told you if I could have located what was needed, a similar phenomenon was present. I have marked, with qq, the two crucial steps in the proof.

A 61-Step Proof Spawning a Useful Method

```
----- Otter 3.3d, April 2004 -----
The process was started by wos on lemma.mcs.anl.gov,
Fri May 21 15:25:43 2004
The command was "otter". The process ID is 15709.

----> UNIT CONFLICT at 2803.60 sec ----> 1061302 [binary,1061301.1,19.1] $ANS(THESIS_3).

Length of proof is 61. Level of proof is 16.

---------------- PROOF ----------------

6 [] -P(i(x,y)) | -P(x) | P(y).
7 [] -P(j(x,y)) | -P(x) | P(y).
9 [] P(i(x,i(y,x))).
10 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
12 [] P(i(x,j(y,x))).
13 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
14 [] P(i(j(j(x,y),x),x)).
15 [] P(i(j(i(x,y),y),j(i(y,x),x))).
16 [] P(j(i(x,y),j(x,y))).
19 [] -P(j(i(B,C),i(j(A,B),j(A,C)))) | $ANS(THESIS_3).
23 [hyper,6,12,12] P(j(x,i(y,j(z,y)))).
24 [hyper,6,9,12] P(i(x,i(y,j(z,y)))).
29 [hyper,6,10,10] P(i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))).
31 [hyper,6,10,24] P(i(i(x,y),i(x,j(z,y)))).
62 [hyper,6,31,13] P(i(j(x,j(y,z)),j(u,j(j(x,y),j(x,z))))).
64 [hyper,6,9,13] P(i(x,i(j(y,j(z,u)),j(j(y,z),j(y,u))))).
82 [hyper,6,10,64] P(i(i(x,j(y,j(z,u))),i(x,j(j(y,z),j(y,u))))).
97 [hyper,6,82,12] P(i(j(x,y),j(j(z,x),j(z,y)))).
103 [hyper,6,9,97] P(i(x,i(j(y,z),j(j(u,y),j(u,z))))).
108 [hyper,6,9,15] P(i(x,i(j(i(y,z),z),j(i(z,y),y)))).
109 [hyper,7,16,97] P(j(j(x,y),j(j(z,x),j(z,y)))).
115 [hyper,7,16,15] P(j(j(i(x,y),y),j(i(y,x),x))).
116 [hyper,7,16,14] P(j(j(j(x,y),x),x)).
117 [hyper,7,16,13] P(j(j(x,j(y,z)),j(j(x,y),j(x,z)))).
118 [hyper,7,16,12] P(j(x,j(y,x))).
120 [hyper,7,16,10] P(j(i(x,i(y,z)),i(i(x,y),i(x,z)))).
121 [hyper,7,16,9] P(j(x,i(y,x))).
```

129 [hyper,6,13,115] P(j(j(j(i(x,y),y),i(y,x)),j(j(i(x,y),y),x))).
141 [hyper,6,12,116] P(j(x,j(j(j(y,z),y),y))).
145 [hyper,6,12,120] P(j(x,j(i(y,i(z,u)),i(i(y,z),i(y,u))))).
156 [hyper,6,12,121] P(j(x,j(y,i(z,y)))).
248 [hyper,7,118,118] P(j(x,j(y,j(z,y)))).
281 [hyper,6,29,64] P(i(i(j(x,j(y,z)),i(j(j(x,y),j(x,z)),u)),i(j(x,j(y,z)),u))).
587 [hyper,6,13,156] P(j(j(x,y),j(x,i(z,y)))).
605 [hyper,6,12,587] P(j(x,j(j(y,z),j(y,i(u,z))))).
1478 [hyper,6,13,141] P(j(j(x,j(j(y,z),y)),j(x,y))).
3236 [hyper,6,29,103] P(i(i(j(x,y),i(j(j(z,x),j(z,y)),u)),i(j(x,y),u))).
3238 [hyper,6,10,103] P(i(i(x,j(y,z)),i(x,j(j(u,y),j(u,z))))).
3695 [hyper,6,10,108] P(i(i(x,j(i(y,z),z)),i(x,j(i(z,y),y)))).
3748 [hyper,6,13,109] P(j(j(j(x,y),j(z,x)),j(j(x,y),j(z,y)))).
4214 [hyper,7,117,117] P(j(j(j(x,j(y,z)),j(x,y)),j(j(x,j(y,z)),j(x,z)))).
4844 [hyper,7,129,23] P(j(j(i(j(x,y),y),y),j(x,y))).
5029 [hyper,6,62,4844] P(j(x,j(j(j(i(j(y,z),z),z),y),j(j(i(j(y,z),z),z),z)))).
5509 [hyper,6,13,145] P(j(j(x,i(y,i(z,u))),j(x,i(i(y,z),i(y,u))))).
14350 [hyper,6,13,605] P(j(j(x,j(y,z)),j(x,j(y,i(u,z))))).
446326 [hyper,6,3695,12] P(i(x,j(i(x,y),y))).
446852 [hyper,6,3238,446326] P(i(x,j(j(y,i(x,z)),j(y,z)))).
446879 [hyper,6,9,446326] P(i(x,i(y,j(i(y,z),z)))).
450897 [hyper,6,10,446879] P(i(i(x,y),i(x,j(i(y,z),z)))).
451088 [hyper,7,16,450897] P(j(i(x,y),i(x,j(i(y,z),z)))).
508507 [hyper,7,4214,248] P(j(j(x,j(j(y,x),z)),j(x,z))).
674213 [hyper,7,508507,5029] P(j(x,j(j(i(j(x,y),y),y),y))).
674542 [hyper,7,14350,674213] P(j(x,j(j(i(j(x,y),y),y),i(z,y)))).
677403 [hyper,7,1478,674542] P(j(x,i(j(x,y),y))).
679992 [hyper,7,587,677403] P(j(x,i(y,i(j(x,z),z)))).
679996 [hyper,7,118,677403] P(j(x,j(y,i(j(y,z),z)))).
688680 [hyper,7,5509,679992] P(j(x,i(i(y,j(x,z)),i(y,z)))).
690596 [hyper,7,679992,118] P(i(x,i(j(j(y,j(z,y)),u),u))).
694664 [hyper,7,688680,688680] P(i(i(x,j(j(y,i(i(z,j(y,u)),i(z,u))),v)),i(x,v))).
697117 [hyper,6,694664,446852] P(i(i(x,j(y,z)),j(y,i(x,z)))).
715220 [hyper,7,3748,679996] P(j(j(i(j(x,y),y),z),j(x,z))).
715475 [hyper,7,715220,451088] P(j(x,i(j(x,y),j(i(y,z),z)))).
715962 [hyper,7,688680,715475] P(i(i(x,j(j(y,i(j(y,z),j(i(z,u),u))),v)),i(x,v))).
718042 [hyper,6,715962,446852] P(i(j(x,y),j(x,j(i(y,z),z)))).
776076 [hyper,6,3236,690596] P(i(j(j(x,y),z),j(y,z))).
776335 [hyper,7,121,776076] P(i(x,i(j(j(y,z),u),j(z,u)))).
qq922929 [hyper,6,281,776335] P(i(j(x,j(y,z)),j(y,j(x,z)))).
qq1055771 [hyper,7,121,922929] P(i(x,i(j(y,j(z,u)),j(z,j(y,u))))).
1056432 [hyper,6,10,1055771] P(i(i(x,j(y,j(z,u))),i(x,j(z,j(y,u))))).
1059748 [hyper,6,1056432,718042] P(i(j(x,y),j(i(y,z),j(x,z)))).
1061301 [hyper,6,697117,1059748] P(j(i(x,y),i(j(z,x),j(z,y)))).

Do you have any suggestions for finding a proof with the desired properties, avoiding both *A*3 and *A*6 totally? The given proof offers a small clue.

You and I can reason through this together, as I offer three subproblems to address. First (subproblem), *A*6 must be blocked when and if it is deduced; it is not in the input, having been commented out. Demodulation will suffice for that. Second (subproblem), the child of *A*6 just given may be crucial to completing the desired proof of Thesis 3. Therefore, a means must be found, if possible, to deduce that child (formula) in such a way that *A*6 *does not* participate. In that a solution to this subproblem is at the heart of the method being offered, I shall turn to the third and last item, which seems straightforward (more or less).

Third (subproblem), with a proof of the child in hand, a means must be found to rely on that (sub)proof to complete the desired total proof, of Thesis 3, a proof exhibiting all of the sought-after properties. For the third subproblem, the approach that appears very promising is to take the input file that produced the almost-satisfying proof of Thesis 3—except for containing two unwanted deduced steps, flagged earlier with qq—, and modify it. The appropriate demodulator would be adjoined, to prevent *A*6 from participating as a deduced step; and, to rely on what would (I hoped) occur from a solution to the second subproblem, the steps of the proof of the desired child would be adjoined as resonators. You and I can now focus on the second subproblem.

The principle underlying the approach I used (and now present) asserts that many proofs to a given conclusion exist. You may have guessed what is about to be written. In the 61-step proof, *A*6 is the fifty-seventh deduced step, and its child is the fifty-eighth. Perhaps OTTER could take the first fifty-six deduced steps and use them, with the input that yielded the 61-step proof, and find another path to the child of *A*6, a path totally free of *A*6. Of course, the idea is to attempt to force or *cram*, as in the cramming strategy, the cited fifty-six steps into the sought-after proof of the charming child. Therefore, the choice is to place the fifty-six steps in the list(sos) of the new and amended input file and rely on a level-saturation (breadth-first) search. The expectation is that, if the sought-after proof of the child is found, most likely a number greater than 1 of steps not among the fifty-six would be needed. Although the following input file is not the one used those years ago (and lost among my files), this file is that which was used in the story being narrated.

An Input File to Deduce, with Appropriate Constraints, a Child

```
set(hyper_res).
assign(max_weight,20).
clear(print_kept).
% set(ancestor_subsume).
set(back_sub).
assign(max_mem,600000).
% assign(max_seconds,7).
% assign(report,900).
% assign(pick_given_ratio,2).
assign(max_proofs,-1).
% set(order_history).
% set(input_sos_first).
set(sos_queue).
set(order_history).
assign(max_distinct_vars,6).
assign(heat,0).

% Modifications to strategy
% Clauses
list(demodulators).
% (P(i(i(i(x,y),x),x)) = junk). % A3
(P(i(j(x,j(y,z)),j(y,j(x,z)))) = junk). % A6
(i(x,junk) = junk).
(i(junk,x) = junk).
(j(x,junk) = junk).
(j(junk,x) = junk).
(P(junk) = $T).
end_of_list.

weight_list(pick_and_purge).
% Following is a 50-step proof of the join, from temp.spinks1.out1j7.
```

```
weight(P(i(x,i(y,j(z,y)))),2).
weight(P(i(i(x,y),i(x,j(z,y)))),2).
weight(P(i(i(i(x,y),x),i(i(x,y),j(z,y)))),2).
weight(P(i(x,i(i(i(y,z),y),i(i(y,z),j(u,z))))),2).
weight(P(i(i(x,i(i(y,z),y)),i(x,i(i(y,z),j(u,z))))),2).
weight(P(i(x,i(i(x,y),j(z,y)))),2).
weight(P(i(x,i(j(y,j(z,u)),j(j(y,z),j(y,u))))),2).
weight(P(i(i(x,j(y,j(z,u))),i(x,j(j(y,z),j(y,u))))),2).
weight(P(i(x,i(j(y,j(z,u)),j(z,j(y,u))))),2).
weight(P(i(j(x,y),j(j(z,x),j(z,y)))),2).
weight(P(i(i(x,j(y,j(z,u))),i(x,j(z,j(y,u))))),2).
weight(P(j(j(i(x,y),y),j(i(y,x),x))),2).
weight(P(j(j(j(x,y),x),x)),2).
weight(P(j(i(x,i(y,z)),i(i(x,y),i(x,z)))),2).
weight(P(j(x,i(y,x))),2).
weight(P(j(x,j(i(x,y),y))),2).
weight(P(j(i(x,y),j(j(i(y,x),x),y))),2).
weight(P(j(j(x,j(j(y,z),y)),j(x,y))),2).
weight(P(j(j(x,i(y,i(z,u))),j(x,i(i(y,z),i(y,u))))),2).
weight(P(j(j(x,y),j(x,i(z,y)))),2).
weight(P(j(j(x,y),j(x,j(i(y,z),z)))),2).
weight(P(j(j(i(j(x,y),y),y),j(x,y))),2).
weight(P(j(j(x,j(y,z)),j(x,j(y,i(u,z))))),2).
weight(P(j(x,j(i(i(y,x),z),z))),2).
weight(P(j(x,j(j(i(j(x,y),y),y),y))),2).
weight(P(j(i(i(x,y),z),j(y,z))),2).
weight(P(j(x,j(j(i(j(x,y),y),y),i(z,y)))),2).
weight(P(j(i(i(x,j(y,z)),y),y)),2).
weight(P(j(x,i(j(x,y),y))),2).
weight(P(j(i(x,i(y,j(x,z))),i(y,j(x,z)))),2).
weight(P(j(x,j(i(i(j(x,y),y),z),z))),2).
weight(P(j(x,i(y,i(j(x,z),z)))),2).
weight(P(i(i(x,y),j(x,y))),2).
weight(P(i(x,j(i(x,y),y))),2).
weight(P(j(i(i(j(x,y),y),z),j(x,z))),2).
weight(P(j(x,i(i(y,j(x,z)),i(y,z)))),2).
weight(P(i(x,i(y,j(i(y,z),z)))),2).
weight(P(i(x,j(y,j(i(x,z),z)))),2).
weight(P(i(i(x,j(j(y,i(i(z,j(y,u)),i(z,u))),v)),i(x,v))),2).
weight(P(i(i(x,y),i(x,j(i(y,z),z)))),2).
weight(P(i(x,j(j(y,i(x,z)),j(y,z)))),2).
weight(P(j(x,i(j(x,y),j(i(y,z),z)))),2).
weight(P(i(i(x,j(y,z)),j(y,i(x,z)))),2).
weight(P(i(i(x,j(j(y,i(j(y,z),j(i(z,u),u))),v)),i(x,v))),2).
weight(P(j(j(x,y),i(j(y,z),j(x,z)))),2).
weight(P(i(j(x,y),j(x,j(i(y,z),z)))),2).
weight(P(i(j(j(x,y),z),j(i(x,y),z))),2).
weight(P(i(j(x,y),j(i(y,z),j(x,z)))),2).
weight(P(j(i(x,y),i(j(y,z),j(x,z)))),2).
weight(P(j(i(x,y),i(j(z,x),j(z,y)))),2).
end_of_list.

list(usable).
```

```
-P(i(x,y)) | -P(x) | P(y).              % Modus
-P(j(x,y)) | -P(x) | P(y).              % Modus
-P(j(i(A,B),i(j(B,C),j(A,C)))) | -P(j(i(B,C),i(j(A,B),j(A,C)))) | $ANS(THESIS_23).
end_of_list.

list(sos).
%  Axioms
P(i(x,i(y,x))).                         % (A1)
P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).     % (A2)
P(i(i(i(x,y),x),x)).                    % (A3)
P(i(x,j(y,x))).                         % (A4)
P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).     % (A5)
%  P(i(j(x,j(y,z)),j(y,j(x,z)))).       % (A6)
P(i(j(j(x,y),x),x)).                    % (A7)
P(i(j(i(x,y),y),j(i(y,x),x))).          % (A8)
P(j(i(x,y),j(x,y))).                    % (A9)
%  Following 56 are initial segment of a proof of Thesis 3 in which A6 occurs
%  as a deduced clause, and where A6 has but one child in the proof of Thesis1, 61-step?
P(j(x,i(y,j(z,y)))).
P(i(x,i(y,j(z,y)))).
P(i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))).
P(i(i(x,y),i(x,j(z,y)))).
P(i(j(x,j(y,z)),j(u,j(j(x,y),j(x,z))))).
P(i(x,i(j(y,j(z,u)),j(j(y,z),j(y,u))))).
P(i(i(x,j(y,j(z,u))),i(x,j(j(y,z),j(y,u))))).
P(i(j(x,y),j(j(z,x),j(z,y)))).
P(i(x,i(j(y,z),j(j(u,y),j(u,z))))).
P(i(x,i(j(i(y,z),z),j(i(z,y),y)))).
P(j(j(x,y),j(j(z,x),j(z,y)))).
P(j(j(i(x,y),y),j(i(y,x),x))).
P(j(j(j(x,y),x),x)).
P(j(j(x,j(y,z)),j(j(x,y),j(x,z)))).
P(j(x,j(y,x))).
P(j(i(x,i(y,z)),i(i(x,y),i(x,z)))).
P(j(x,i(y,x))).
P(j(j(j(i(x,y),y),i(y,x)),j(j(i(x,y),y),x))).
P(j(x,j(j(j(y,z),y),y))).
P(j(x,j(i(y,i(z,u)),i(i(y,z),i(y,u))))).
P(j(x,j(y,i(z,y)))).
P(j(x,j(y,j(z,y)))).
P(i(i(j(x,j(y,z)),i(j(j(x,y),j(x,z)),u)),i(j(x,j(y,z)),u))).
P(j(j(x,y),j(x,i(z,y)))).
P(j(x,j(j(y,z),j(y,i(u,z))))).
P(j(j(x,j(j(y,z),y)),j(x,y))).
P(i(i(j(x,y),i(j(j(z,x),j(z,y)),u)),i(j(x,y),u))).
P(i(i(x,j(y,z)),i(x,j(j(u,y),j(u,z))))).
P(i(i(x,j(i(y,z),z)),i(x,j(i(z,y),y)))).
P(j(j(j(x,y),j(z,x)),j(j(x,y),j(z,y)))).
P(j(j(j(x,j(y,z)),j(x,y)),j(j(x,j(y,z)),j(x,z)))).
P(j(j(i(j(x,y),y),y),j(x,y))).
P(j(x,j(j(j(i(j(y,z),z),z),y),j(j(i(j(y,z),z),z),z)))).
P(j(j(x,i(y,i(z,u))),j(x,i(i(y,z),i(y,u))))).
P(j(j(x,j(y,z)),j(x,j(y,i(u,z))))).
```

```
P(i(x,j(i(x,y),y))).
P(i(x,j(j(y,i(x,z)),j(y,z)))).
P(i(x,i(y,j(i(y,z),z)))).
P(i(i(x,y),i(x,j(i(y,z),z)))).
P(j(i(x,y),i(x,j(i(y,z),z)))).
P(j(j(x,j(j(y,x),z)),j(x,z))).
P(j(x,j(j(i(j(x,y),y),y),y))).
P(j(x,j(j(i(j(x,y),y),y),i(z,y)))).
P(j(x,i(j(x,y),y))).
P(j(x,i(y,i(j(x,z),z)))).
P(j(x,j(y,i(j(y,z),z)))).
P(j(x,i(i(y,j(x,z)),i(y,z)))).
P(i(x,i(j(j(y,j(z,y)),u),u))).
P(i(i(x,j(j(y,i(i(z,j(y,u)),i(z,u))),v)),i(x,v))).
P(i(i(x,j(y,z)),j(y,i(x,z)))).
P(j(j(i(j(x,y),y),z),j(x,z))).
P(j(x,i(j(x,y),j(i(y,z),z)))).
P(i(i(x,j(j(y,i(j(y,z),j(i(z,u),u))),v)),i(x,v))).
P(i(j(x,y),j(x,j(i(y,z),z)))).
P(i(j(j(x,y),z),j(y,z))).
P(i(x,i(j(j(y,z),u),j(z,u)))).
end_of_list.

list(passive).
-P(i(a1,i(j(a2,j(a3,a4)),j(a3,j(a2,a4))))) | $ANS(child).
-P(i(i(A,B),j(A,B))) | $ANS(THESIS_1).                 % Lemma
-P(j(i(A,B),i(j(B,C),j(A,C)))) | $ANS(THESIS_2).       % Lemma
-P(j(i(B,C),i(j(A,B),j(A,C)))) | $ANS(THESIS_3).       % Lemma
end_of_list.

list(hot).
-P(i(x,y)) | -P(x) | P(y).              % Modus
P(i(i(x,y),j(x,y))).
end_of_list.
```

All went according to plan; indeed, this first run succeeded, producing the following 5-step proof of the child, of course relying on far more than axioms 1, 2, 4, 5, 7, 8, and 9.

A 5-Step Proof of a Most Wanted Child

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on elephant.mcs.anl.gov,
Thu May 28 14:25:22 2009
The command was "otter".  The process ID is 10840.

----> UNIT CONFLICT at  44.73 sec ----> 40189 [binary,40188.1,74.1] $ANS(child).

Length of proof is 5.  Level of proof is 3.

---------------- PROOF ----------------

7 [] -P(i(x,y)) | -P(x) | P(y).
8 [] -P(j(x,y)) | -P(x) | P(y).
```

22 [] P(i(j(x,j(y,z)),j(u,j(j(x,y),j(x,z))))).
24 [] P(i(i(x,j(y,j(z,u))),i(x,j(j(y,z),j(y,u))))).
31 [] P(j(j(x,j(y,z)),j(j(x,y),j(x,z)))).
39 [] P(j(x,j(y,j(z,y)))).
41 [] P(j(j(x,y),j(x,i(z,y)))).
67 [] P(i(i(x,j(y,z)),j(y,i(x,z)))).
74 [] -P(i(a1,i(j(a2,j(a3,a4)),j(a3,j(a2,a4))))) | $ANS(child).
125 [hyper,7,24,22] P(i(j(x,j(y,z)),j(j(u,j(x,y)),j(u,j(x,z))))).
174 [hyper,8,31,41] P(j(j(j(x,y),x),j(j(x,y),i(z,y)))).
1387 [hyper,7,67,125] P(j(j(x,j(y,z)),i(j(y,j(z,u)),j(x,j(y,u))))).
2060 [hyper,8,174,39] P(j(j(j(x,j(y,x)),z),i(u,z))).
40188 [hyper,8,2060,1387] P(i(x,i(j(y,j(z,u)),j(z,j(y,u))))).

In other words, the child in question now has different parents from those it had in the 61-step proof. (For clarity, you see that a number of items, outside the original nine-axiom set, are present at the input level in the given 5-step proof, items arising from being included in the amended set of support list.) To obtain the desired proof of Thesis 3 within the constraints in focus, you take the input file, move the fifty-six steps from list(sos) to just after weight_list(pick_and_purge) while turning them into resonators, each with an assigned value of -2). You follow these fifty-six immediately with five resonators, corresponding to the five deduced steps of the proof just given, each assigned a value of -1. Then, also important, you comment out set(sos_queue), to avoid a breadth-first search, and comment in assign(pick_given_ratio,2), to instruct the program to mix complexity preference with level saturation. With the value 2 assigned to the pick_given_ratio, a strategy formulated by McCune, the program chooses (for directing the reasoning) two items based on complexity, 1 based on first come first serve, 2, 1, and so on. After I did these things, I expected to receive from OTTER a proof of Thesis 3 in which neither *A*3 nor *A*6 would participate in any way. The first proof presented to me was that of the child, a proof of length 42. Then came a 40-step proof of Thesis 2, a 54-step proof of Thesis 3, a 57-step proof of the join of 2 and 3, and a 61-step proof of Thesis 1. Each of the proofs relied on, as axioms, 1, 2, 4, 5, 7, 8, and 9. If you are puzzled about the lengths just quoted, I am fairly sure that the explanation rests with the inclusion of the sixty-one resonators with their assigned values. Such inclusions often dramatically modify results when compared with earlier results. More important than the constrained proof of Thesis 3, as well as the other cited proofs, was what was implied by the early detailed occurrences of this current story. Specifically, *A*6 was proved, as a new theorem, to be dependent on 1, 2, 4, 5, 7, 8, and 9; by way of amplification, *A*6 had occurred as a deduced step in a study in which the axioms consisted of but seven of the original nine. So, with the proof (found by OTTER) that contains *A*6 as a deduced step by relying on axioms 1, 2, 4, 5, 7, 8, and 9 as hypotheses, the dependency of *A*6, viewed as a new theorem, can be classified as a discovery for automated reasoning, as was the case for *A*3. I shall present later in this notebook, in Section 6, a satisfying proof establishing *A*6 dependent. And thus the second story ends happily.

Before turning to the next story, a few observations are in order. The approach just given would have merited use even if *A*6 had been the parent of more than one formula that followed its derivation. Iteration would be the way to proceed. You would proceed as I did but now with the negation of the first child of *A*6 placed in list(passive) with the goal of obtaining the needed proof that culminates with the derivation of the first child and without allowing *A*6 to participate in any manner. Then you would amend further the list(sos) with the new proof steps (that led to the derivation of the first child of *A*6 without *A*6 participating), as well as proof steps of the original proof preceding the second child and not dependent on *A*6, and would use as target the second child, placing its negation in list(passive), now with the goal of deriving the second child and with the given constraints. You would proceed in this manner, gathering proof steps along the way, until the last child of *A*6 was proved. Of course, the method under discussion is useful when the goal is to avoid any unwanted formula or equation and replace its role by other formulas or equations.

With 3 and 6, among 1 through 9, now proved dependent, how could I resist seeking other dependencies among the remaining seven! And a third story begins to unfold. I chose *A*7 as a candidate to be proved dependent on the set consisting of 1, 2, 4, 5, 8, and 9. That choice was a rather natural choice in that *A*7 resembles somewhat *A*3, and the latter is a dependent axiom. Two approaches merit consideration, a direct one and an indirect one. In the former, the negation of *A*7 is placed in the passive list, and, in list(sos) are placed 1, 2, 4, 5, 8, and 9. In addition, two demodulators are included to prevent the retention of either *A*3 or *A*6 when and if either is deduced. The object is to complete a proof deducing *A*7 from the cited six axioms of *9BCSK*, showing *A*7 to be dependent. For the indirect approach, all is the same as in the direct approach except that, rather than the negation of *A*7 being placed in list(passive), you place negations of, say, Theses 1, 2, and 3. After numerous experiments with each approach, I had learned nothing; no proofs of interest had been completed.

My colleague Z. Ernst came to the rescue—or perhaps rescue is the wrong word in that I would have preferred *A*7 to be dependent—finding the following three-element model (with Mace4, see the Web ww.mcs.anl.gov/AR/mace4) showing that *A*7 is in fact independent of the six.

```
-------- Model 1 at 0.01 seconds --------

 a : 1

 b : 2

 i :
    | 0 1 2
  --+------
  0 | 0 1 2
  1 | 0 0 2
  2 | 0 1 0

 j :
    | 0 1 2
  --+------
  0 | 0 1 2
  1 | 0 0 2
  2 | 0 0 0

 P :
      0 1 2
   ---------
      1 0 0

-------- end of model --------
```

Although *A*7 was proved by Ernst to be independent of the other original nine axioms given to me by Spinks, not all was lost. After all, I had found proofs in which both *A*3 and *A*6 were totally absent, proofs of Theses 1, 2, and 3, and the join of 2 and 3. So, just perhaps, I could behave as if *A*7 were dependent in the sense that it was not needed, in any way, for various proofs of theorems of interest to Spinks. On May 26, 2004, OTTER sought (on my request) and found proofs of each of the three theses and of the conjunction of 2 and 3, proofs in which 3, 6, and 7 are totally absent. Later in Section 6 of this notebook, I shall present pleasing proofs of Thesis 1 and of the conjunction of 2 and 3, where the hypotheses consisted of axioms 1, 2, 4, 5, 8, and 9. By the way, for the curious, the 3-literal clause for condensed detachment for the function *i* is dependent on the 3-literal clause for *j* in the presence of axioms 1, 2, 4, 5, 8, and 9. The proof comes quickly, in two steps, which I leave to you.

4. An Extension of the BCSK Logic

One extension of the *BCSK* logic, which was of interest to Spinks, is obtained by adjoining the following axiom, *A*10.

```
P(i(j(j(x,y),y),j(j(y,x),x))).  % A10
```

An interesting theorem to prove is captured, in its negated form, with the following clause for *A*10a; the formula, *A*10a, to be proved is equivalent to *A*10, and the two appropriate proofs found by OTTER totally avoid *A*3, *A*6, and *A*7.

-P(i(j(A,B),i(A,B))) | $ANS(thm).

An appropriate move to test the power of the abbreviated axiom system, now consisting of seven axioms with the cited addition of *A*10, is to give OTTER an input file whose initial set of support consists of the seven axioms. As part of the experiment, the demodulator list should contain equalities that, respectively, block the retention of *A*3, *A*6, and *A*7 if and when each is deduced. Just perhaps, *A*7 might now be dependent on the seven-axiom system. From Veroff, I had in hand a 42-step proof of the theorem under consideration to initiate the study, a proof that does depend on the three axioms I intended to avoid (at both the axiomatic and deduced levels). The original goal was to shorten that proof. More pertinent from the viewpoint of this notebook, I sought to find a proof establishing each of *A*3, *A*6, and *A*7 to be totally unneeded.

All went smoothly as shown with the following 23-step proof, a proof obtained in my original study with Spinks in 2004 and 2005.

A 23-Step Proof of the Alternative to A10

```
----- Otter 3.3d, April 2004 -----
The process was started by wos on jaguar.mcs.anl.gov,
Mon Jun 28 08:55:32 2004
The command was "otter". The process ID is 28971.
----> UNIT CONFLICT at   0.04 sec ----> 309 [binary,308.1,19.1] $ANS(thm).

Length of proof is 23. Level of proof is 13.

---------------- PROOF ----------------

19 [] -P(i(j(A,B),i(A,B)))  $ANS(thm).
22 [] -P(i(x,y)) | -P(x) | P(y).
23 [] -P(j(x,y)) | -P(x) | P(y).
25 [] P(i(x,i(y,x))).
26 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
27 [] P(i(x,j(y,x))).
28 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
29 [] P(i(j(i(x,y),y),j(i(y,x),x))).
30 [] P(j(i(x,y),j(x,y))).
31 [] P(i(j(j(x,y),y),j(j(y,x),x))).
34 [hyper,22,25,27] P(i(x,i(y,j(z,y)))).
39 [hyper,23,30,25] P(j(x,i(y,x))).
47 [hyper,22,26,26] P(i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))).
48 [hyper,22,25,26] P(i(x,i(i(y,i(z,u)),i(i(y,z),i(y,u))))).
50 [hyper,22,26,25] P(i(i(x,y),i(x,x))).
58 [hyper,22,26,48] P(i(i(x,i(y,i(z,u))),i(x,i(i(y,z),i(y,u))))).
63 [hyper,22,47,50] P(i(i(x,i(x,y)),i(x,y))).
80 [hyper,22,58,25] P(i(i(x,y),i(i(z,x),i(z,y)))).
82 [hyper,23,30,63] P(j(i(x,i(x,y)),i(x,y))).
92 [hyper,22,26,80] P(i(i(i(x,y),i(z,x)),i(i(x,y),i(z,y)))).
107 [hyper,22,29,82] P(j(i(i(x,y),x),x)).
116 [hyper,22,92,34] P(i(i(j(x,y),z),i(y,z))).
120 [hyper,23,30,116] P(j(i(j(x,y),z),i(y,z))).
127 [hyper,22,116,28] P(i(j(x,y),j(j(z,x),j(z,y)))).
```

135 [hyper,22,127,107] P(j(j(x,i(i(y,z),y)),j(x,y))).
137 [hyper,22,127,39] P(j(j(x,y),j(x,i(z,y)))).
141 [hyper,22,116,31] P(i(x,j(j(x,y),y))).
176 [hyper,23,135,120] P(j(i(j(x,i(y,z)),y),y)).
244 [hyper,22,29,176] P(j(i(x,j(y,i(x,z))),j(y,i(x,z)))).
263 [hyper,23,244,141] P(j(j(x,i(x,y)),i(x,y))).
287 [hyper,22,127,263] P(j(j(x,j(y,i(y,z))),j(x,i(y,z)))).
297 [hyper,23,287,137] P(j(j(x,y),i(x,y))).
308 [hyper,23,297,297] P(i(j(x,y),i(x,y))).

An examination of this 23-step proof reveals a perhaps odd occurrence. In particular, the added axiom, *A*10, is used as a parent just once. If I recall correctly (about what occurred those five years ago), I thought that, in an odd sense, this sparse use of *A*10 might imply that *A*7 was dependent in the *BCSK*+ logic. Why might this conjecture be made? Well, although the Ernst model established independence from the set consisting of 1, 2, 4, 5, 8, and 9, *A*7 was shown to be unneeded when seeking proofs of various theorems. Also, as demonstrated, *A*7 was unneeded in the *BCSK*+ logic for proving the two theorems establishing the equivalence of *A*10 with *A*10a. Still, I fear I have not supplied much of a clue; therefore, chalk it up to untamed intuition.

Regardless of justification, I decided, after a short while, to seek a proof showing *A*7 dependent in *BCSK*+, where *A*3 and *A*6 were not allowed to participate in any way. By way of review, I simply took an input file similar to that given earlier, commented out in list(sos) each of *A*3, *A*6, and *A*7, and included demodulators to block retention of 3 and 6 when and if either was deduced. OTTER won this game, presenting to me the following 24-step proof establishing *A*7 dependent in the *BCSK*+ logic. Of course, *A*10 was placed in list(sos).

A 24-Step Proof of the Dependence of A7 in BCSK+

----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on jaguar.mcs.anl.gov,
Tue Mar 8 16:10:03 2005
The command was "otter". The process ID is 4337.
----> UNIT CONFLICT at 0.13 sec ----> 679 [binary,678.1,17.1] $ANS(a7).

Length of proof is 24. Level of proof is 11.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] -P(j(x,y)) | -P(x) | P(y).
5 [] P(i(x,i(y,x))).
6 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
7 [] P(i(x,j(y,x))).
8 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
9 [] P(i(j(i(x,y),y),j(i(y,x),x))).
10 [] P(j(i(x,y),j(x,y))).
11 [] P(i(j(j(x,y),y),j(j(y,x),x))).
17 [] -P(i(j(j(a1,a2),a1),a1)) | $ANS(a7).
57 [hyper,1,6,6] P(i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))).
58 [hyper,1,5,6] P(i(x,i(i(y,i(z,u)),i(i(y,z),i(y,u))))).
59 [hyper,1,6,5] P(i(i(x,y),i(x,x))).
61 [hyper,1,5,7] P(i(x,i(y,j(z,y)))).
69 [hyper,2,10,5] P(j(x,i(y,x))).
80 [hyper,1,6,58] P(i(i(x,i(y,i(z,u))),i(x,i(i(y,z),i(y,u))))).

```
83 [hyper,1,57,59] P(i(i(x,i(x,y)),i(x,y))).
93 [hyper,1,7,69] P(j(x,j(y,i(z,y)))).
124 [hyper,1,80,5] P(i(i(x,y),i(i(z,x),i(z,y)))).
127 [hyper,2,10,83] P(j(i(x,i(x,y)),i(x,y))).
136 [hyper,1,8,93] P(j(j(x,y),j(x,i(z,y)))).
188 [hyper,1,6,124] P(i(i(i(x,y),i(z,x)),i(i(x,y),i(z,y)))).
198 [hyper,1,9,127] P(j(i(i(x,y),x),x)).
288 [hyper,1,188,61] P(i(i(j(x,y),z),i(y,z))).
344 [hyper,1,288,11] P(i(x,j(j(x,y),y))).
395 [hyper,2,10,344] P(j(x,j(j(x,y),y))).
398 [hyper,1,124,344] P(i(i(x,y),i(x,j(j(y,z),z)))).
550 [hyper,1,8,395] P(j(j(x,j(x,y)),j(x,y))).
564 [hyper,1,398,288] P(i(i(j(x,y),z),j(j(i(y,z),u),u))).
615 [hyper,1,11,550] P(j(j(j(x,y),x),x)).
618 [hyper,2,198,564] P(j(j(i(x,y),x),x)).
632 [hyper,2,136,615] P(j(j(j(x,y),x),i(z,x))).
650 [hyper,1,11,618] P(j(j(x,i(x,y)),i(x,y))).
678 [hyper,2,650,632] P(i(j(j(x,y),x),x)).
```

Still on the subject of determining whether *A*7 is independent or dependent in *BCSK*+, I now realize as I write that no need existed for attempting to find an appropriate proof. Indeed, an inspection of the output files that led to the deduction of *A*10a would have sufficed. When I did make this inspection on May 31, 2009, and searched the file for the formula (*A*7), in one of the output files, I found it among what OTTER calls the given clauses. The given clauses are those OTTER chooses to initiate applications of the inference rules in use. In other words, although no proof establishing the dependence had been found, such an inspection would have answered the question in the affirmative, with irrefutable evidence that *A*7 is dependent. This type of inspection is not one I had recourse to earlier, from what I can recall, but it is clearly useful, especially when the knowledge is the goal, even without a formal proof.

5. A More Intriguing Extension of the BCSK Logic

Those many years ago, Spinks asked me to find proofs in an extension of *BCSK* far more elaborate than is *BCSK*+, namely, *SBPC*. To obtain *SBPC* from *BCSK*, you adjoin the following six axioms (in clause notation), where the function *a* denotes logical **and** and the function *o* denotes logical **or**.

```
P(j(x,o(x,y))).                      % A11
P(i(y,o(x,y))).                      % A12
P(j(j(x,z),j(j(y,z),j(o(x,y),z)))).  % A13
P(i(a(x,y),x)).                      % A14
P(j(a(x,y),y)).                      % A15
P(i(i(x,y),i(i(x,z),i(x,a(y,z))))).  % A16
```

Spinks requested short proofs, if possible, of the following four theorems, each expressed in its negated form.

```
-P(j(i(A,B),i(o(A,C),o(B,C)))) | $ANS(1).
-P(j(i(A,B),i(o(C,A),o(C,B)))) | $ANS(2).
-P(j(i(A,B),j(i(B,A),i(a(A,C),a(B,C))))) | $ANS(3).
-P(j(i(A,B),i(a(C,A),a(C,B)))) | $ANS(4).
```

I began my study with proofs again supplied by Veroff.

As predictable, rather than the full set of fifteen axioms, I was intent upon conducting my experiments without relying on *A*3, *A*6, or *A*7 in any way. Of course, I could not be certain that none of the three would be needed at the deduced level. I, therefore, included in the list(demodulators) corresponding demodulators to discard, if and when deduced, each of the three cited so-called unwanted items.

At the moment, I cannot recall which of the four theorems presented a problem; I do know, from my notes, that at least one of them could not be proved, under the given constraints, until I returned to history. What saved the day was a proof I had obtained in an earlier study, a proof that deduced a (former) child of *a*3, a proof in which *A*3 was totally absent. My notes assert that the proof I turned to has length 92. When I say turned to an earlier proof, in general I mean its deduced steps are used as resonators or are used as *hints*. The hints strategy is due to Veroff. Whereas a resonator focuses on expressions similar to it with all variables treated as indistinguishable, a hint focuses (usually) on items that are identical to, subsumed by, or subsume it.

Success eventually was the result. OTTER returned a 53-step proof of the first of the four theorems, a 64-step proof of the second, a 103-step proof of the third (found during the writing of this notebook), and a 94-step proof of the fourth (found during the writing of this notebook); see Section 6 for proofs. Many experiments were required, as well as much use of refinement methodology detailed in the book *Automated Reasoning and the Discovery of Missing and Elegant Proofs* by Wos and Pieper. The last significant reductions in proof length (of the proofs of the third and fourth theorems) were obtained by heavy reliance on cramming. Briefly, OTTER was given proofs of steps near the end of the proofs in hand and asked to (in effect) force their proofs steps into (I hoped) shorter proofs of the targets.

If you are somewhat engrossed in the story now being told, your curiosity might suggest two items for thought. First, with the knowledge that *A*7 was proved dependent in *BCSK*+ (which was obtained by merely adjoining one more axiom), you might conjecture that *A*7 is dependent in *SBPC*. After all, six axioms were adjoined to obtain *SBPc* from *BCSK*. Second, short of an explicit proof, you might wonder about finding *A*7 to be dependent simply by browsing in an output file that was produced while I was studying one of the four theorems. Specifically, you might wonder if, among the given clauses (those chosen to drive the reasoning), you might find *A*7 present. Of course, the output file must be the result of using an input file satisfying certain properties, properties that include the omission of *A*7 from list(sos) and the omission from its correspondent in list(demodulators). About the latter, the idea is *not* to block the retention of *A*7 when and if it is deduced. The input file in focus had best also have both *A*3 and *A*6 omitted from the list(sos) and blocked, with appropriate demodulators, from retention if and when deduced.

With regard to the second question—and a pause before I return to the past—at this writing, I conducted the appropriate experiment. I chose an input file that had been used those years ago to seek a proof of the first of the four cited theorems of interest to Spinks. I made minor modifications to satisfy the given constraints and instructed OTTER to seek a proof of the theorem in focus. I was, of course, not interested in that proof but, instead, curious about finding, if such occurred, *A*7 among the given clauses. I did find it there, thus proving that *A*7 is dependent on a set of axioms, namely, 1, 2, 4, 5, 8, 9, and the six in the functions *a* and *o*. That experiment did not yield an explicit proof of the given dependence, but the output file answered in the affirmative the question under consideration, establishing the conjecture to be a good one. So, now I return to history and to the actions I took those years ago.

The discovery (those years ago) that *A*7 is dependent in the *BCSK*+ logic (obtained by adding *A*10 to the original nine axioms, then removing any use of *A*3 and *A*6) led me to consider the possibility that that formula is dependent in this second extension of the *BCSK* logic. Indeed, would it not be more than piquant to find that *A*7 is independent in the original study and then find it dependent in two extensions of the logic? And, as the following proof shows—the shortest so far discovered—that is exactly what was found.

A 35-Step Proof of the Dependence of A7 in SBPC

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on theorem.mcs.anl.gov,
Sun Mar 20 12:31:56 2005
The command was "otter".  The process ID is 20352.
----> UNIT CONFLICT at   0.09 sec ----> 904 [binary,903.1,24.1] $ANS(a7).
```

Length of proof is 35. Level of proof is 20.

---------------- PROOF ----------------

10 [] -P(i(x,y)) | -P(x) | P(y).
11 [] -P(j(x,y)) | -P(x) | P(y).
12 [] P(i(x,i(y,x))).
13 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
14 [] P(i(x,j(y,x))).
15 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
16 [] P(i(j(i(x,y),y),j(i(y,x),x))).
17 [] P(j(i(x,y),j(x,y))).
18 [] P(j(x,o(x,y))).
19 [] P(i(y,o(x,y))).
20 [] P(j(j(x,z),j(j(y,z),j(o(x,y),z)))).
24 [] -P(i(j(j(a1,a2),a1),a1)) | $ANS(a7).
130 [hyper,10,13,13] P(i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))).
133 [hyper,10,12,14] P(i(x,i(y,j(z,y)))).
135 [hyper,10,12,15] P(i(x,i(j(y,j(z,u)),j(j(y,z),j(y,u))))).
138 [hyper,11,17,16] P(j(j(i(x,y),y),j(i(y,x),x))).
139 [hyper,11,17,15] P(j(j(x,j(y,z)),j(j(x,y),j(x,z)))).
140 [hyper,11,17,14] P(j(x,j(y,x))).
142 [hyper,11,17,12] P(j(x,i(y,x))).
180 [hyper,10,130,133] P(i(i(x,i(j(y,x),z)),i(x,z))).
197 [hyper,11,140,140] P(j(x,j(y,j(z,y)))).
244 [hyper,10,180,135] P(i(j(x,y),j(j(z,x),j(z,y)))).
285 [hyper,11,17,244] P(j(j(x,y),j(j(z,x),j(z,y)))).
290 [hyper,10,244,142] P(j(j(x,y),j(x,i(z,y)))).
298 [hyper,10,15,285] P(j(j(j(x,y),j(z,x)),j(j(x,y),j(z,y)))).
347 [hyper,11,298,197] P(j(j(j(x,y),z),j(y,z))).
362 [hyper,11,285,347] P(j(j(x,j(j(y,z),u)),j(x,j(z,u)))).
371 [hyper,11,347,138] P(j(x,j(i(x,y),y))).
416 [hyper,11,362,139] P(j(j(x,j(y,z)),j(y,j(x,z)))).
449 [hyper,11,285,371] P(j(j(x,y),j(x,j(i(y,z),z)))).
488 [hyper,11,416,416] P(j(x,j(j(y,j(x,z)),j(y,z)))).
506 [hyper,11,416,138] P(j(i(x,y),j(j(i(y,x),x),y))).
559 [hyper,11,449,18] P(j(x,j(i(o(x,y),z),z))).
584 [hyper,11,139,488] P(j(j(x,j(y,j(x,z))),j(x,j(y,z)))).
603 [hyper,11,506,19] P(j(j(i(o(x,y),y),y),o(x,y))).
604 [hyper,11,506,14] P(j(j(i(j(x,y),y),y),j(x,y))).
657 [hyper,11,584,559] P(j(x,j(i(o(x,y),j(x,z)),z))).
681 [hyper,11,416,657] P(j(i(o(x,y),j(x,z)),j(x,z))).
698 [hyper,11,603,681] P(o(x,j(x,y))).
709 [hyper,11,488,698] P(j(j(x,j(o(y,j(y,z)),u)),j(x,u))).
727 [hyper,11,709,140] P(j(x,x)).
738 [hyper,11,20,727] P(j(j(x,y),j(o(y,x),y))).
767 [hyper,11,709,738] P(j(j(j(x,y),x),x)).
814 [hyper,11,285,767] P(j(j(x,j(j(y,z),y)),j(x,y))).
845 [hyper,11,814,604] P(j(j(i(j(j(x,y),x),x),x),x)).
851 [hyper,11,814,290] P(j(j(j(i(x,y),z),y),i(x,y))).
903 [hyper,11,851,845] P(i(j(j(x,y),x),x)).

For the curious, the first study yielded a 39-step proof, a proof that the usual methods were unable to improve upon. However, with a most unsophisticated form of cramming, the given 35-step proof was found. In particular, rather than relying on a subproof of one of the late steps, OTTER was merely given the first 34 steps of the 39-step proof and told to apply level saturation. In other words, no attention was paid to the possible presence of steps among the thirty-four that were not used in the proof of the thirty-fourth step.

6. Additional Pleasing Proofs

In this section, I offer various proofs, many or all of which were promised earlier. You will see that, although this notebook is being written in 2009, I borrow from research conducted years ago. If proofs are not your wish, you might prefer simply to turn to Section 7, the last in this notebook. On the other hand, as is so typical of my notebooks, each of the proofs offers an implied challenge. Indeed, each is the shortest, of its class (within the constraints given), I have found so far as of June 1, 2009.

First I give two proofs that, respectively, establish $A3$ and $A6$ to be dependent on axioms 1, 2, 4, 5, 8, and 9.

A 14-Step Proof Showing A3 Dependent in the BCSK logic

```
----- Otter 3.3d, April 2004 -----
The process was started by wos on jaguar.mcs.anl.gov,
Thu May 27 10:43:03 2004
The command was "otter".  The process ID is 31886.
----> UNIT CONFLICT at   0.02 sec ----> 150 [binary,149.1,17.1] $ANS(a3).

Length of proof is 14.  Level of proof is 10.

---------------- PROOF ----------------

6 [] -P(i(x,y)) | -P(x) | P(y).
7 [] -P(j(x,y)) | -P(x) | P(y).
9 [] P(i(x,i(y,x))).
10 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
11 [] P(i(x,j(y,x))).
12 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
13 [] P(i(j(i(x,y),y),j(i(y,x),x))).
14 [] P(j(i(x,y),j(x,y))).
17 [] -P(i(i(i(a1,a2),a1),a1)) | $ANS(a3).
24 [hyper,6,9,9] P(i(x,i(y,i(z,y)))).
38 [hyper,6,10,10] P(i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))).
40 [hyper,6,10,24] P(i(i(x,y),i(x,i(z,y)))).
41 [hyper,6,10,9] P(i(i(x,y),i(x,x))).
46 [hyper,7,14,40] P(j(i(x,y),i(x,i(z,y)))).
58 [hyper,6,38,41] P(i(i(x,i(x,y)),i(x,y))).
97 [hyper,7,14,58] P(j(i(x,i(x,y)),i(x,y))).
115 [hyper,6,13,97] P(j(i(i(x,y),x),x)).
120 [hyper,6,11,115] P(j(x,j(i(i(y,z),y),y))).
124 [hyper,6,12,120] P(j(j(x,i(i(y,z),y)),j(x,y))).
129 [hyper,7,124,46] P(j(i(i(i(x,y),z),y),i(x,y))).
138 [hyper,7,124,129] P(j(i(i(i(i(x,y),x),z),x),x)).
144 [hyper,6,13,138] P(j(i(x,i(i(i(x,y),x),z)),i(i(i(x,y),x),z))).
149 [hyper,7,144,9] P(i(i(i(x,y),x),x)).
```

A 23-Step Proof Showing A6 to Be Dependent in the BCSK Logic

----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on lemma.mcs.anl.gov,
Wed Jun 15 12:19:27 2005
The following has proofs of lengths 23.
----> UNIT CONFLICT at 1.04 sec ----> 2169 [binary,2168.1,18.1] $ANS(A6).

Length of proof is 23. Level of proof is 12.

---------------- PROOF ----------------

9 [] -P(i(x,y)) | -P(x) | P(y).
10 [] -P(j(x,y)) | -P(x) | P(y).
12 [] P(i(x,i(y,x))).
13 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
14 [] P(i(x,j(y,x))).
15 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
16 [] P(i(j(i(x,y),y),j(i(y,x),x))).
17 [] P(j(i(x,y),j(x,y))).
18 [] -P(i(j(a1,j(a2,a3)),j(a2,j(a1,a3)))) | $ANS(A6).
53 [hyper,9,14,14] P(j(x,i(y,j(z,y)))).
54 [hyper,9,12,14] P(i(x,i(y,j(z,y)))).
55 [hyper,9,14,12] P(j(x,i(y,i(z,y)))).
65 [hyper,9,13,54] P(i(i(x,y),i(x,j(z,y)))).
75 [hyper,10,17,14] P(j(x,j(y,x))).
80 [hyper,10,75,75] P(j(x,j(y,j(z,y)))).
86 [hyper,9,65,15] P(i(j(x,j(y,z)),j(u,j(j(x,y),j(x,z))))).
88 [hyper,9,12,15] P(i(x,i(j(y,j(z,u)),j(j(y,z),j(y,u))))).
96 [hyper,9,13,88] P(i(i(x,j(y,j(z,u))),i(x,j(j(y,z),j(y,u))))).
98 [hyper,9,96,86] P(i(j(x,j(y,z)),j(j(u,j(x,y)),j(u,j(x,z))))).
106 [hyper,10,17,16] P(j(j(i(x,y),y),j(i(y,x),x))).
116 [hyper,9,15,106] P(j(j(j(i(x,y),y),i(y,x)),j(j(i(x,y),y),x))).
120 [hyper,10,116,55] P(j(j(i(i(x,y),y),y),i(x,y))).
121 [hyper,10,116,53] P(j(j(i(j(x,y),y),y),j(x,y))).
129 [hyper,9,15,121] P(j(j(j(i(j(x,y),y),y),x),j(j(i(j(x,y),y),y),y))).
145 [hyper,10,129,80] P(j(j(i(j(j(x,j(y,x)),z),z),z),z)).
156 [hyper,10,75,145] P(j(x,j(j(i(j(j(y,j(z,y)),u),u),u),u))).
163 [hyper,9,15,156] P(j(j(x,j(i(j(j(y,j(z,y)),u),u),u)),j(x,u))).
168 [hyper,10,163,17] P(j(i(i(j(j(x,j(y,x)),z),z),z),z)).
2139 [hyper,10,120,168] P(i(j(j(x,j(y,x)),z),z)).
2157 [hyper,9,12,2139] P(i(x,i(j(j(y,j(z,y)),u),u))).
2161 [hyper,9,13,2157] P(i(i(x,j(j(y,j(z,y)),u)),i(x,u))).
2168 [hyper,9,2161,98] P(i(j(x,j(y,z)),j(y,j(x,z)))).

A 24-Step Proof of Thesis 1 in BCSK

----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on jaguar.mcs.anl.gov,
Sun May 8 17:07:27 2005
The command was "otter". The process ID is 3642.

The following has proofs of lengths 28 27 25 24.

----> UNIT CONFLICT at 951.00 sec ----> 29757 [binary,29756.1,48.1]
$ANS(THESIS_1). Length of proof is 24. Level of proof is 11.

---------------- PROOF ----------------

9 [] -P(i(x,y))| -P(x)|P(y).
10 [] -P(j(x,y))| -P(x)|P(y).
12 [] P(i(x,i(y,x))).
13 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
14 [] P(i(x,j(y,x))).
15 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
16 [] P(i(j(i(x,y),y),j(i(y,x),x))).
17 [] P(j(i(x,y),j(x,y))).
48 [] -P(i(i(A,B),j(A,B)))|$ANS(THESIS_1).
103 [hyper,9,12,14] P(i(x,i(y,j(z,y)))).
105 [hyper,9,14,12] P(j(x,i(y,i(z,y)))).
109 [hyper,10,17,16] P(j(j(i(x,y),y),j(i(y,x),x))).
111 [hyper,10,17,15] P(j(j(x,j(y,z)),j(j(x,y),j(x,z)))).
113 [hyper,10,17,14] P(j(x,j(y,x))).
117 [hyper,9,14,17] P(j(x,j(i(y,z),j(y,z)))).
134 [hyper,9,13,103] P(i(i(x,y),i(x,j(z,y)))).
140 [hyper,9,15,109] P(j(j(j(i(x,y),y),i(y,x)),j(j(i(x,y),y),x))).
144 [hyper,10,111,111] P(j(j(j(x,j(y,z)),j(x,y)),j(j(x,j(y,z)),j(x,z)))).
148 [hyper,9,15,113] P(j(j(x,y),j(x,x))).
163 [hyper,9,15,117] P(j(j(x,i(y,z)),j(x,j(y,z)))).
228 [hyper,10,140,105] P(j(j(i(i(x,y),y),y),i(x,y))).
238 [hyper,10,144,117] P(j(j(i(x,y),j(j(x,y),z)),j(i(x,y),z))).
245 [hyper,10,144,148] P(j(j(x,j(x,y)),j(x,y))).
420 [hyper,10,163,228] P(j(j(i(i(x,y),y),y),j(x,y))).
479 [hyper,10,113,245] P(j(x,j(j(y,j(y,z)),j(y,z)))).
976 [hyper,9,14,420] P(j(x,j(j(i(i(y,z),z),z),j(y,z)))).
1160 [hyper,10,111,479] P(j(j(x,j(y,j(y,z))),j(x,j(y,z)))).
2744 [hyper,10,238,976] P(j(i(i(i(x,y),y),y),j(x,y))).
5987 [hyper,10,1160,2744] P(j(i(i(i(x,j(x,y)),j(x,y)),j(x,y)),j(x,y))).
9328 [hyper,10,228,5987] P(i(i(x,j(x,y)),j(x,y))).
14429 [hyper,9,12,9328] P(i(x,i(i(y,j(y,z)),j(y,z)))).
21281 [hyper,9,13,14429] P(i(i(x,i(y,j(y,z))),i(x,j(y,z)))).
29756 [hyper,9,21281,134] P(i(i(x,y),j(x,y))).

A 45-Step Proof of the Join of Theses 2 and 3 in BCSK

----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on jaguar.mcs.anl.gov,
Thu May 19 13:27:52 2005
The command was "otter". The process ID is 26865.

The following has proofs of lengths 45 52.
-----> EMPTY CLAUSE at 0.08 sec ----> 651 [hyper,11,615,590] $ANS(THESIS_23).
Length of proof is 45. Level of proof is 18.

---------------- PROOF ----------------

9 [] -P(i(x,y))| -P(x)|P(y).
10 [] -P(j(x,y))| -P(x)|P(y).
11 [] -P(j(i(A,B),i(j(B,C),j(A,C))))| -P(j(i(B,C),i(j(A,B),j(A,C))))|$ANS(THESIS_23).
12 [] P(i(x,i(y,x))).
13 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
14 [] P(i(x,j(y,x))).
15 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
16 [] P(i(j(i(x,y),y),j(i(y,x),x))).
17 [] P(j(i(x,y),j(x,y))).
71 [hyper,9,14,12] P(j(x,i(y,i(z,y)))).
74 [hyper,10,17,14] P(j(x,j(y,x))).
83 [hyper,9,13,13] P(i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))).
84 [hyper,9,12,13] P(i(x,i(i(y,i(z,u)),i(i(y,z),i(y,u))))).
85 [hyper,9,13,12] P(i(i(x,y),i(x,x))).
90 [hyper,9,13,84] P(i(i(x,i(y,i(z,u))),i(x,i(i(y,z),i(y,u))))).
94 [hyper,9,83,85] P(i(i(x,i(x,y)),i(x,y))).
98 [hyper,9,85,14] P(i(x,x)).
103 [hyper,9,90,12] P(i(i(x,y),i(i(z,x),i(z,y)))).
111 [hyper,10,17,94] P(j(i(x,i(x,y)),i(x,y))).
122 [hyper,10,17,103] P(j(i(x,y),i(i(z,x),i(z,y)))).
129 [hyper,9,103,15] P(i(i(x,j(y,j(z,u))),i(x,j(j(y,z),j(y,u))))).
130 [hyper,9,103,14] P(i(i(x,y),i(x,j(z,y)))).
136 [hyper,10,17,16] P(j(j(i(x,y),y),j(i(y,x),x))).
137 [hyper,9,103,16] P(i(i(x,j(i(y,z),z)),i(x,j(i(z,y),y)))).
140 [hyper,9,16,111] P(j(i(i(x,y),x),x)).
152 [hyper,9,129,14] P(i(j(x,y),j(j(z,x),j(z,y)))).
155 [hyper,10,17,130] P(j(i(x,y),i(x,j(z,y)))).
188 [hyper,9,137,14] P(i(x,j(i(x,y),y))).
197 [hyper,9,103,152] P(i(i(x,j(y,z)),i(x,j(j(u,y),j(u,z))))).
200 [hyper,9,152,140] P(j(j(x,i(i(y,z),y)),j(x,y))).
201 [hyper,9,152,136] P(j(j(x,j(i(y,z),z)),j(x,j(i(z,y),y)))).
202 [hyper,9,152,122] P(j(j(x,i(y,z)),j(x,i(i(u,y),i(u,z))))).
247 [hyper,9,197,188] P(i(x,j(j(y,i(x,z)),j(y,z)))).
250 [hyper,9,197,15] P(i(j(x,j(y,z)),j(j(u,j(x,y)),j(u,j(x,z))))).
252 [hyper,10,200,155] P(j(i(i(j(x,y),z),y),j(x,y))).
279 [hyper,10,202,71] P(j(x,i(i(y,z),i(y,i(u,z))))).
284 [hyper,10,17,247] P(j(x,j(j(y,i(x,z)),j(y,z)))).
312 [hyper,9,250,252] P(j(j(x,j(i(i(j(y,z),u),z),y)),j(x,j(i(i(j(y,z),u),z),z)))).
335 [hyper,9,250,284] P(j(j(x,j(y,j(z,i(y,u)))),j(x,j(y,j(z,u))))).
342 [hyper,10,312,74] P(j(x,j(i(i(j(x,y),z),y),y))).
350 [hyper,10,335,74] P(j(j(x,i(y,z)),j(y,j(x,z)))).
352 [hyper,10,201,342] P(j(x,j(i(y,i(j(x,y),z)),i(j(x,y),z)))).
385 [hyper,10,350,279] P(j(i(x,y),j(z,i(x,i(u,y))))).
405 [hyper,9,250,352] P(j(j(x,j(y,i(z,i(j(y,z),u)))),j(x,j(y,i(j(y,z),u))))).
419 [hyper,10,405,385] P(j(i(x,y),j(z,i(j(z,x),y)))).
441 [hyper,10,419,98] P(j(x,i(j(x,y),y))).
456 [hyper,10,202,441] P(j(x,i(i(y,j(x,z)),i(y,z)))).
497 [hyper,10,350,456] P(j(i(x,j(y,z)),j(y,i(x,z)))).
524 [hyper,10,497,247] P(j(j(x,i(y,z)),i(y,j(x,z)))).
525 [hyper,10,497,152] P(j(j(x,y),i(j(y,z),j(x,z)))).
556 [hyper,9,152,524] P(j(j(x,j(y,i(z,u))),j(x,i(z,j(y,u))))).
587 [hyper,9,152,525] P(j(j(x,j(y,z)),j(x,i(j(z,u),j(y,u))))).
590 [hyper,10,556,419] P(j(i(x,y),i(j(z,x),j(z,y)))).

615 [hyper,10,587,17] P(j(i(x,y),i(j(y,z),j(x,z)))).

I now switch from *BCSK* to *BCSK+*.

An 18-Step Proof Showing A7 to Be Dependent in BCSK+

```
----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on lemma.mcs.anl.gov,
Fri Jun 10 10:57:07 2005
The command was "otter".  The process ID is 27094.
----> UNIT CONFLICT at   0.04 sec ----> 345 [binary,344.1,17.1] $ANS(a7).

Length of proof is 18.  Level of proof is 8.

---------------- PROOF ----------------

1 [] -P(i(x,y)) | -P(x) | P(y).
2 [] -P(j(x,y)) | -P(x) | P(y).
5 [] P(i(x,i(y,x))).
7 [] P(i(x,j(y,x))).
8 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
9 [] P(i(j(i(x,y),y),j(i(y,x),x))).
10 [] P(j(i(x,y),j(x,y))).
11 [] P(i(j(j(x,y),y),j(j(y,x),x))).
17 [] -P(i(j(j(a1,a2),a1),a1)) | $ANS(a7).
53 [hyper,1,7,7] P(j(x,i(y,j(z,y)))).
58 [hyper,2,10,9] P(j(j(i(x,y),y),j(i(y,x),x))).
59 [hyper,2,10,8] P(j(j(x,j(y,z)),j(j(x,y),j(x,z)))).
60 [hyper,2,10,7] P(j(x,j(y,x))).
62 [hyper,2,10,5] P(j(x,i(y,x))).
76 [hyper,1,8,58] P(j(j(j(i(x,y),y),i(y,x)),j(j(i(x,y),y),x))).
82 [hyper,2,59,59] P(j(j(j(x,j(y,z)),j(x,y)),j(j(x,j(y,z)),j(x,z)))).
85 [hyper,1,8,60] P(j(j(x,y),j(x,x))).
89 [hyper,1,7,62] P(j(x,j(y,i(z,y)))).
111 [hyper,2,76,53] P(j(j(i(j(x,y),y),y),j(x,y))).
113 [hyper,2,82,85] P(j(j(x,j(x,y)),j(x,y))).
127 [hyper,1,8,89] P(j(j(x,y),j(x,i(z,y)))).
149 [hyper,1,11,113] P(j(j(j(x,y),x),x)).
317 [hyper,2,60,149] P(j(x,j(j(j(y,z),y),y))).
323 [hyper,2,59,317] P(j(j(x,j(j(y,z),y)),j(x,y))).
331 [hyper,2,323,127] P(j(j(j(i(x,y),z),y),i(x,y))).
332 [hyper,2,323,111] P(j(j(i(j(x,y),x),x),x)).
344 [hyper,2,331,332] P(i(j(j(x,y),x),x)).
```

An 18-step Proof Deducing A10 in BCSK+

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----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on lemma.mcs.anl.gov,
Tue Mar  8 15:11:26 2005
The command was "otter".  The process ID is 1181.
----> UNIT CONFLICT at   3.12 sec ----> 12089 [binary,12088.1,14.1] $ANS(a10).
```

Length of proof is 18. Level of proof is 13.

---------------- PROOF ----------------

14 [] -P(i(j(j(a1,a2),a2),j(j(a2,a1),a1))) | $ANS(a10).
17 [] -P(i(x,y)) | -P(x) | P(y).
20 [] P(i(x,i(y,x))).
21 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
24 [] P(i(j(i(x,y),y),j(i(y,x),x))).
25 [] P(j(i(x,y),j(x,y))).
26 [] P(i(j(x,y),i(x,y))).
38 [hyper,17,20,21] P(i(x,i(i(y,i(z,u)),i(i(y,z),i(y,u))))).
43 [hyper,17,21,38] P(i(i(x,i(y,i(z,u))),i(x,i(i(y,z),i(y,u))))).
56 [hyper,17,43,20] P(i(i(x,y),i(i(z,x),i(z,y)))).
68 [hyper,17,21,56] P(i(i(i(x,y),i(z,x)),i(i(x,y),i(z,y)))).
88 [hyper,17,20,26] P(i(x,i(j(y,z),i(y,z)))).
90 [hyper,17,26,25] P(i(i(x,y),j(x,y))).
109 [hyper,17,68,88] P(i(i(i(x,y),z),i(j(x,y),z))).
116 [hyper,17,20,90] P(i(x,i(i(y,z),j(y,z)))).
125 [hyper,17,109,109] P(i(j(i(x,y),z),i(j(x,y),z))).
147 [hyper,17,68,116] P(i(i(j(x,y),z),i(i(x,y),z))).
150 [hyper,17,21,116] P(i(i(x,i(y,z)),i(x,j(y,z)))).
166 [hyper,17,56,125] P(i(i(x,j(i(y,z),u)),i(x,i(j(y,z),u)))).
212 [hyper,17,166,24] P(i(j(i(x,y),y),i(j(y,x),x))).
226 [hyper,17,147,212] P(i(i(i(x,y),y),i(j(y,x),x))).
242 [hyper,17,56,226] P(i(i(x,i(i(y,z),z)),i(x,i(j(z,y),y)))).
271 [hyper,17,242,147] P(i(i(j(x,y),y),i(j(y,x),x))).
288 [hyper,17,109,271] P(i(j(j(x,y),y),i(j(y,x),x))).
12088 [hyper,17,150,288] P(i(j(j(x,y),y),j(j(y,x),x))).

A 23-Step Proof Deducing A10a in BCSK+

----- Otter 3.3d, April 2004 -----
The process was started by wos on jaguar.mcs.anl.gov,
Mon Jun 28 08:55:32 2004
The command was "otter". The process ID is 28971.
----> UNIT CONFLICT at   0.04 sec ----> 309 [binary,308.1,19.1] $ANS(thm).

Length of proof is 23. Level of proof is 13.

---------------- PROOF ----------------

19 [] -P(i(j(A,B),i(A,B))) | $ANS(thm).
22 [] -P(i(x,y)) | -P(x) | P(y).
23 [] -P(j(x,y)) | -P(x) | P(y).
25 [] P(i(x,i(y,x))).
26 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
27 [] P(i(x,j(y,x))).
28 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
29 [] P(i(j(i(x,y),y),j(i(y,x),x))).
30 [] P(j(i(x,y),j(x,y))).
31 [] P(i(j(j(x,y),y),j(j(y,x),x))).

34 [hyper,22,25,27] P(i(x,i(y,j(z,y)))).
39 [hyper,23,30,25] P(j(x,i(y,x))).
47 [hyper,22,26,26] P(i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))).
48 [hyper,22,25,26] P(i(x,i(i(y,i(z,u)),i(i(y,z),i(y,u))))).
50 [hyper,22,26,25] P(i(i(x,y),i(x,x))).
58 [hyper,22,26,48] P(i(i(x,i(y,i(z,u))),i(x,i(i(y,z),i(y,u))))).
63 [hyper,22,47,50] P(i(i(x,i(x,y)),i(x,y))).
80 [hyper,22,58,25] P(i(i(x,y),i(i(z,x),i(z,y)))).
82 [hyper,23,30,63] P(j(i(x,i(x,y)),i(x,y))).
92 [hyper,22,26,80] P(i(i(i(x,y),i(z,x)),i(i(x,y),i(z,y)))).
107 [hyper,22,29,82] P(j(i(i(x,y),x),x)).
116 [hyper,22,92,34] P(i(i(j(x,y),z),i(y,z))).
120 [hyper,23,30,116] P(j(i(j(x,y),z),i(y,z))).
127 [hyper,22,116,28] P(i(j(x,y),j(j(z,x),j(z,y)))).
135 [hyper,22,127,107] P(j(j(x,i(i(y,z),y)),j(x,y))).
137 [hyper,22,127,39] P(j(j(x,y),j(x,i(z,y)))).
141 [hyper,22,116,31] P(i(x,j(j(x,y),y))).
176 [hyper,23,135,120] P(j(i(j(x,i(y,z)),y),y)).
244 [hyper,22,29,176] P(j(i(x,j(y,i(x,z))),j(y,i(x,z)))).
263 [hyper,23,244,141] P(j(j(x,i(x,y)),i(x,y))).
287 [hyper,22,127,263] P(j(j(x,j(y,i(y,z))),j(x,i(y,z)))).
297 [hyper,23,287,137] P(j(j(x,y),i(x,y))).
308 [hyper,23,297,297] P(i(j(x,y),i(x,y))).

Finally, I focus on *SBPC*, the perhaps more intriguing extension of *BCSK* Four theorems are of concern. Earlier, I cited proofs of respective lengths of 53, 64, 103, and 94, proofs of four theorems in *SBPC*. For convenience, I repeat their negations here.

-P(j(i(A,B),i(o(A,C),o(B,C)))) | $ANS(1).
-P(j(i(A,B),i(o(C,A),o(C,B)))) | $ANS(2).
-P(j(i(A,B),j(i(B,A),i(a(A,C),a(B,C))))) | $ANS(3).
-P(j(i(A,B),i(a(C,A),a(C,B)))) | $ANS(4).

A 53-Step Proof in SBPC of the First Theorem

----- Otter 3.3f, August 2004 -----
The process was started by wos on jaguar.mcs.anl.gov,
Thu Jan 27 10:47:54 2005
The command was "otter". The process ID is 31295.
----> UNIT CONFLICT at 0.10 sec ----> 863 [binary,862.1,30.1] $ANS(1).

Length of proof is 53. Level of proof is 27.

---------------- PROOF ----------------

16 [] -P(i(x,y)) | -P(x) | P(y).
17 [] -P(j(x,y)) | -P(x) | P(y).
18 [] P(i(x,i(y,x))).
19 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
20 [] P(i(x,j(y,x))).
21 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
22 [] P(i(j(i(x,y),y),j(i(y,x),x))).
23 [] P(j(i(x,y),j(x,y))).

24 [] P(j(x,o(x,y))).
25 [] P(i(y,o(x,y))).
26 [] P(j(j(x,z),j(j(y,z),j(o(x,y),z)))).
30 [] -P(j(i(A,B),i(o(A,C),o(B,C)))) | $ANS(1).
93 [hyper,16,19,19] P(i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))).
96 [hyper,16,18,20] P(i(x,i(y,j(z,y)))).
98 [hyper,16,18,21] P(i(x,i(j(y,j(z,u)),j(j(y,z),j(y,u))))).
101 [hyper,17,23,22] P(j(j(i(x,y),y),j(i(y,x),x))).
102 [hyper,17,23,21] P(j(j(x,j(y,z)),j(j(x,y),j(x,z)))).
103 [hyper,17,23,20] P(j(x,j(y,x))).
118 [hyper,16,18,25] P(i(x,i(y,o(z,y)))).
153 [hyper,16,93,96] P(i(i(x,i(j(y,x),z)),i(x,z))).
169 [hyper,17,103,103] P(j(x,j(y,j(z,y)))).
182 [hyper,16,19,118] P(i(i(x,y),i(x,o(z,y)))).
191 [hyper,16,153,98] P(i(j(x,y),j(j(z,x),j(z,y)))).
229 [hyper,17,23,191] P(j(j(x,y),j(j(z,x),j(z,y)))).
235 [hyper,16,191,24] P(j(j(x,y),j(x,o(y,z)))).
241 [hyper,16,21,229] P(j(j(j(x,y),j(z,x)),j(j(x,y),j(z,y)))).
263 [hyper,17,241,169] P(j(j(j(x,y),z),j(y,z))).
265 [hyper,17,229,263] P(j(j(x,j(j(y,z),u)),j(x,j(z,u)))).
268 [hyper,17,263,241] P(j(j(x,y),j(j(y,z),j(x,z)))).
272 [hyper,17,263,101] P(j(x,j(i(x,y),y))).
279 [hyper,17,265,102] P(j(j(x,j(y,z)),j(y,j(x,z)))).
282 [hyper,17,268,268] P(j(j(j(j(x,y),j(z,y)),u),j(j(z,x),u))).
294 [hyper,17,268,23] P(j(j(j(x,y),z),j(i(x,y),z))).
303 [hyper,17,229,272] P(j(j(x,y),j(x,j(i(y,z),z)))).
325 [hyper,17,279,279] P(j(x,j(j(y,j(x,z)),j(y,z)))).
335 [hyper,17,279,101] P(j(i(x,y),j(j(i(y,x),x),y))).
337 [hyper,17,279,26] P(j(j(x,y),j(j(z,y),j(o(z,x),y)))).
355 [hyper,17,294,235] P(j(i(x,y),j(x,o(y,z)))).
356 [hyper,17,294,229] P(j(i(x,y),j(j(z,x),j(z,y)))).
377 [hyper,17,303,24] P(j(x,j(i(o(x,y),z),z))).
384 [hyper,17,102,325] P(j(j(x,j(y,j(x,z))),j(x,j(y,z)))).
399 [hyper,17,335,25] P(j(j(i(o(x,y),y),y),o(x,y))).
418 [hyper,17,268,355] P(j(j(j(x,o(y,z)),u),j(i(x,y),u))).
440 [hyper,17,356,182] P(j(j(x,i(y,z)),j(x,i(y,o(u,z))))).
446 [hyper,17,356,18] P(j(j(x,y),j(x,i(z,y)))).
482 [hyper,17,384,377] P(j(x,j(i(o(x,y),j(x,z)),z))).
562 [hyper,17,279,482] P(j(i(o(x,y),j(x,z)),j(x,z))).
570 [hyper,17,399,562] P(o(x,j(x,y))).
578 [hyper,17,325,570] P(j(j(x,j(o(y,j(y,z)),u)),j(x,u))).
594 [hyper,17,578,103] P(j(x,x)).
599 [hyper,17,337,594] P(j(j(x,y),j(o(x,y),y))).
604 [hyper,17,26,594] P(j(j(x,y),j(o(y,x),y))).
615 [hyper,17,268,599] P(j(j(j(o(x,y),y),z),j(j(x,y),z))).
630 [hyper,17,578,604] P(j(j(j(x,y),x),x)).
682 [hyper,17,229,630] P(j(j(x,j(j(y,z),y)),j(x,y))).
692 [hyper,17,282,682] P(j(j(j(x,y),z),j(j(z,x),x))).
702 [hyper,17,682,446] P(j(j(j(i(x,y),z),y),i(x,y))).
740 [hyper,17,229,702] P(j(j(x,j(j(i(y,z),u),z)),j(x,i(y,z)))).
752 [hyper,17,282,740] P(j(j(j(i(x,y),z),u),j(j(u,y),i(x,y)))).
760 [hyper,17,752,399] P(j(j(o(x,y),y),i(o(x,y),y))).
767 [hyper,17,615,760] P(j(j(x,y),i(o(x,y),y))).

779 [hyper,17,440,767] P(j(j(x,y),i(o(x,y),o(z,y)))).
819 [hyper,17,692,779] P(j(j(i(o(x,y),o(z,y)),x),x)).
852 [hyper,17,752,819] P(j(j(x,o(y,z)),i(o(x,z),o(y,z)))).
862 [hyper,17,418,852] P(j(i(x,y),i(o(x,z),o(y,z)))).

A 64-Step Proof in SBPC of the Second Theorem

----- Otter 3.3f, August 2004 -----
The process was started by wos on jaguar.mcs.anl.gov,
Wed Jan 26 14:44:18 2005
The command was "otter". The process ID is 10563.
----> UNIT CONFLICT at 0.17 sec ----> 1366 [binary,1365.1,28.1] $ANS(2).

Length of proof is 64. Level of proof is 25.

---------------- PROOF ----------------

13 [] -P(i(x,y)) | -P(x) | P(y).
14 [] -P(j(x,y)) | -P(x) | P(y).
15 [] P(i(x,i(y,x))).
16 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
17 [] P(i(x,j(y,x))).
18 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
19 [] P(i(j(i(x,y),y),j(i(y,x),x))).
20 [] P(j(i(x,y),j(x,y))).
21 [] P(j(x,o(x,y))).
22 [] P(i(y,o(x,y))).
23 [] P(j(j(x,z),j(j(y,z),j(o(x,y),z)))).
28 [] -P(j(i(A,B),i(o(C,A),o(C,B)))) | $ANS(2).
111 [hyper,13,15,15] P(i(x,i(y,i(z,y)))).
112 [hyper,13,16,16] P(i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))).
113 [hyper,13,15,16] P(i(x,i(i(y,i(z,u)),i(i(y,z),i(y,u))))).
116 [hyper,13,15,17] P(i(x,i(y,j(z,y)))).
118 [hyper,13,15,18] P(i(x,i(j(y,j(z,u)),j(j(y,z),j(y,u))))).
121 [hyper,14,20,19] P(j(j(i(x,y),y),j(i(y,x),x))).
122 [hyper,14,20,18] P(j(j(x,j(y,z)),j(j(x,y),j(x,z)))).
123 [hyper,14,20,17] P(j(x,j(y,x))).
125 [hyper,14,20,15] P(j(x,i(y,x))).
136 [hyper,14,20,22] P(j(x,o(y,x))).
144 [hyper,14,23,21] P(j(j(x,o(y,z)),j(o(y,x),o(y,z)))).
166 [hyper,13,112,111] P(i(i(x,i(i(y,x),z)),i(x,z))).
174 [hyper,13,112,116] P(i(i(x,i(j(y,x),z)),i(x,z))).
190 [hyper,14,123,123] P(j(x,j(y,j(z,y)))).
192 [hyper,14,23,123] P(j(j(x,j(y,z)),j(o(z,x),j(y,z)))).
231 [hyper,13,166,113] P(i(i(x,y),i(i(z,x),i(z,y)))).
242 [hyper,13,174,118] P(i(j(x,y),j(j(z,x),j(z,y)))).
269 [hyper,13,16,231] P(i(i(i(x,y),i(z,x)),i(i(x,y),i(z,y)))).
273 [hyper,13,231,22] P(i(i(x,y),i(x,o(z,y)))).
278 [hyper,14,20,242] P(j(j(x,y),j(j(z,x),j(z,y)))).
282 [hyper,13,242,136] P(j(j(x,y),j(x,o(z,y)))).
283 [hyper,13,242,125] P(j(j(x,y),j(x,i(z,y)))).
290 [hyper,13,269,111] P(i(i(i(x,y),z),i(y,z))).

295 [hyper,14,20,273] P(j(i(x,y),i(x,o(z,y)))).
320 [hyper,13,18,278] P(j(j(j(x,y),j(z,x)),j(j(x,y),j(z,y)))).
363 [hyper,14,283,21] P(j(x,i(y,o(x,z)))).
408 [hyper,13,290,269] P(i(i(x,y),i(i(y,z),i(x,z)))).
437 [hyper,14,320,190] P(j(j(j(x,y),z),j(y,z))).
477 [hyper,14,20,408] P(j(i(x,y),i(i(y,z),i(x,z)))).
520 [hyper,14,278,437] P(j(j(x,j(j(y,z),u)),j(x,j(z,u)))).
528 [hyper,14,437,320] P(j(j(x,y),j(j(y,z),j(x,z)))).
534 [hyper,14,437,121] P(j(x,j(i(x,y),y))).
554 [hyper,14,520,122] P(j(j(x,j(y,z)),j(y,j(x,z)))).
578 [hyper,14,528,282] P(j(j(j(x,o(y,z)),u),j(j(x,z),u))).
597 [hyper,14,278,534] P(j(j(x,y),j(x,j(i(y,z),z)))).
631 [hyper,14,554,554] P(j(x,j(j(y,j(x,z)),j(y,z)))).
644 [hyper,14,554,283] P(j(x,j(j(x,y),i(z,y)))).
651 [hyper,14,554,121] P(j(i(x,y),j(j(i(y,x),x),y))).
653 [hyper,14,554,23] P(j(j(x,y),j(j(z,y),j(o(z,x),y)))).
724 [hyper,14,597,21] P(j(x,j(i(o(x,y),z),z))).
735 [hyper,14,122,631] P(j(j(x,j(y,j(x,z))),j(x,j(y,z)))).
759 [hyper,14,192,644] P(j(o(i(x,y),z),j(j(z,y),i(x,y)))).
804 [hyper,14,651,22] P(j(j(i(o(x,y),y),y),o(x,y))).
810 [hyper,14,653,295] P(j(j(x,i(y,o(z,u))),j(o(x,i(y,u)),i(y,o(z,u))))).
892 [hyper,14,735,724] P(j(x,j(i(o(x,y),j(x,z)),z))).
909 [hyper,14,810,363] P(j(o(x,i(y,z)),i(y,o(x,z)))).
947 [hyper,14,554,892] P(j(i(o(x,y),j(x,z)),j(x,z))).
961 [hyper,14,278,909] P(j(j(x,o(y,i(z,u))),j(x,i(z,o(y,u))))).
1008 [hyper,14,804,947] P(o(x,j(x,y))).
1024 [hyper,14,631,1008] P(j(j(x,j(o(y,j(y,z)),u)),j(x,u))).
1044 [hyper,14,1024,144] P(j(j(j(x,y),o(x,z)),o(x,z))).
1046 [hyper,14,1024,123] P(j(x,x)).
1075 [hyper,14,578,1044] P(j(j(j(x,y),z),o(x,z))).
1089 [hyper,14,653,1046] P(j(j(x,y),j(o(x,y),y))).
1127 [hyper,14,1075,804] P(o(i(o(x,y),y),o(x,y))).
1168 [hyper,14,528,1089] P(j(j(j(o(x,y),y),z),j(j(x,y),z))).
1189 [hyper,14,759,1127] P(j(j(o(x,y),y),i(o(x,y),y))).
1209 [hyper,14,1168,1189] P(j(j(x,y),i(o(x,y),y))).
1245 [hyper,14,528,1209] P(j(j(i(o(x,y),y),z),j(j(x,y),z))).
1269 [hyper,14,1245,477] P(j(j(x,y),i(i(y,z),i(o(x,y),z)))).
1291 [hyper,14,1075,1269] P(o(x,i(i(y,z),i(o(x,y),z)))).
1321 [hyper,14,909,1291] P(i(i(x,y),o(z,i(o(z,x),y)))).
1340 [hyper,14,20,1321] P(j(i(x,y),o(z,i(o(z,x),y)))).
1365 [hyper,14,961,1340] P(j(i(x,y),i(o(z,x),o(z,y)))).

A 103-Step Proof in SBPC of the Third Theorem

----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on octopus.mcs.anl.gov,
Wed Jun 3 11:10:19 2009
The command was "otter". The process ID is 30561.

----> UNIT CONFLICT at   0.21 sec ----> 5098 [binary,5097.1,28.1] $ANS(3).

Length of proof is 103. Level of proof is 36.

---------------- PROOF ----------------

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28 [] -P(j(i(A,B),j(i(B,A),i(a(A,C),a(B,C)))))|$ANS(3).
134 [] -P(i(x,y))| -P(x)|P(y).
135 [] -P(j(x,y))| -P(x)|P(y).
136 [] P(i(x,i(y,x))).
137 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
138 [] P(i(x,j(y,x))).
139 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
140 [] P(i(j(i(x,y),y),j(i(y,x),x))).
141 [] P(j(i(x,y),j(x,y))).
142 [] P(j(x,o(x,y))).
143 [] P(i(x,o(y,x))).
144 [] P(j(j(x,y),j(j(z,y),j(o(x,z),y)))).
145 [] P(i(a(x,y),x)).
146 [] P(j(a(x,y),y)).
147 [] P(i(i(x,y),i(i(x,z),i(x,a(y,z))))).
148 [hyper,134,136,136] P(i(x,i(y,i(z,y)))).
149 [hyper,134,137,137] P(i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))).
150 [hyper,134,136,137] P(i(x,i(i(y,i(z,u)),i(i(y,z),i(y,u))))).
153 [hyper,134,136,138] P(i(x,i(y,j(z,y)))).
157 [hyper,134,136,139] P(i(x,i(j(y,j(z,u)),j(j(y,z),j(y,u))))).
160 [hyper,135,141,140] P(j(j(i(x,y),y),j(i(y,x),x))).
161 [hyper,135,141,139] P(j(j(x,j(y,z)),j(j(x,y),j(x,z)))).
162 [hyper,135,141,138] P(j(x,j(y,x))).
228 [hyper,134,149,148] P(i(i(x,i(i(y,x),z)),i(x,z))).
237 [hyper,134,149,153] P(i(i(x,i(j(y,x),z)),i(x,z))).
241 [hyper,134,137,153] P(i(i(x,y),i(x,j(z,y)))).
268 [hyper,135,162,162] P(j(x,j(y,j(z,y)))).
284 [hyper,134,228,150] P(i(i(x,y),i(i(z,x),i(z,y)))).
287 [hyper,134,228,136] P(i(x,x)).
296 [hyper,134,237,157] P(i(j(x,y),j(j(z,x),j(z,y)))).
301 [hyper,135,141,241] P(j(i(x,y),i(x,j(z,y)))).
306 [hyper,134,137,241] P(i(i(i(x,y),x),i(i(x,y),j(z,y)))).
345 [hyper,134,137,284] P(i(i(i(x,y),i(z,x)),i(i(x,y),i(z,y)))).
357 [hyper,135,141,287] P(j(x,x)).
358 [hyper,134,147,287] P(i(i(x,y),i(x,a(x,y)))).
365 [hyper,135,141,296] P(j(j(x,y),j(j(z,x),j(z,y)))).
374 [hyper,134,296,146] P(j(j(x,a(y,z)),j(x,z))).
422 [hyper,134,345,148] P(i(i(i(x,y),z),i(y,z))).
425 [hyper,135,144,357] P(j(j(x,y),j(o(y,x),y))).
476 [hyper,134,139,365] P(j(j(j(x,y),j(z,x)),j(j(x,y),j(z,y)))).
485 [hyper,134,296,374] P(j(j(x,j(y,a(z,u))),j(x,j(y,u)))).
503 [hyper,134,422,358] P(i(x,i(y,a(y,x)))).
504 [hyper,134,422,345] P(i(i(x,y),i(i(y,z),i(x,z)))).
505 [hyper,134,422,306] P(i(x,i(i(x,y),j(z,y)))).
545 [hyper,135,476,268] P(j(j(j(x,y),z),j(y,z))).
639 [hyper,135,141,504] P(j(i(x,y),i(i(y,z),i(x,z)))).
668 [hyper,134,504,145] P(i(i(x,y),i(a(x,z),y))).
673 [hyper,135,141,505] P(j(x,i(i(x,y),j(z,y)))).
724 [hyper,135,365,545] P(j(j(x,j(j(y,z),u)),j(x,j(z,u)))).
730 [hyper,135,545,476] P(j(j(x,y),j(j(y,z),j(x,z)))).
733 [hyper,135,545,160] P(j(x,j(i(x,y),y))).
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822 [hyper,135,724,161] P(j(j(x,j(y,z)),j(y,j(x,z)))).
829 [hyper,135,730,730] P(j(j(j(j(x,y),j(z,y)),u),j(j(z,x),u))).
860 [hyper,135,730,141] P(j(j(j(x,y),z),j(i(x,y),z))).
883 [hyper,135,730,733] P(j(j(j(i(x,y),y),z),j(x,z))).
889 [hyper,135,365,733] P(j(j(x,y),j(x,j(i(y,z),z)))).
948 [hyper,135,822,822] P(j(x,j(j(y,j(x,z)),j(y,z)))).
967 [hyper,135,822,160] P(j(i(x,y),j(j(i(y,x),x),y))).
1020 [hyper,135,860,365] P(j(i(x,y),j(j(z,x),j(z,y)))).
1041 [hyper,135,829,883] P(j(j(x,i(y,z)),j(y,j(x,z)))).
1113 [hyper,135,889,142] P(j(x,j(i(o(x,y),z),z))).
1133 [hyper,135,161,948] P(j(j(x,j(y,j(x,z))),j(x,j(y,z)))).
1217 [hyper,135,967,145] P(j(j(i(x,a(x,y)),a(x,y)),x)).
1218 [hyper,135,967,143] P(j(j(i(o(x,y),y),y),o(x,y))).
1220 [hyper,135,967,138] P(j(j(i(j(x,y),y),y),j(x,y))).
1262 [hyper,135,1020,136] P(j(j(x,y),j(x,i(z,y)))).
1282 [hyper,135,1041,639] P(j(i(x,y),j(i(z,x),i(z,y)))).
1391 [hyper,135,1133,1113] P(j(x,j(i(o(x,y),j(x,z)),z))).
1450 [hyper,135,822,1220] P(j(x,j(j(i(j(x,y),y),y),y))).
1566 [hyper,135,1282,136] P(j(i(x,y),i(x,i(z,y)))).
1579 [hyper,135,822,1391] P(j(i(o(x,y),j(x,z)),j(x,z))).
1764 [hyper,135,1218,1579] P(o(x,j(x,y))).
1787 [hyper,135,948,1764] P(j(j(x,j(o(y,j(y,z)),u)),j(x,u))).
1823 [hyper,135,1787,425] P(j(j(j(x,y),x),x)).
1849 [hyper,135,365,1823] P(j(j(x,j(j(y,z),y)),j(x,y))).
1892 [hyper,135,1849,1262] P(j(j(j(i(x,y),z),y),i(x,y))).
1919 [hyper,135,829,1892] P(j(j(x,i(y,j(x,z))),i(y,j(x,z)))).
1924 [hyper,135,365,1892] P(j(j(x,j(j(i(y,z),u),z)),j(x,i(y,z)))).
1947 [hyper,135,1919,673] P(i(i(x,y),j(x,y))).
1960 [hyper,135,829,1924] P(j(j(j(i(x,y),z),u),j(j(u,y),i(x,y)))).
1974 [hyper,135,1924,1450] P(j(x,i(j(x,y),y))).
2021 [hyper,134,284,1947] P(i(i(x,i(y,z)),i(x,j(y,z)))).
2031 [hyper,135,829,1960] P(j(j(x,i(y,z)),j(j(j(x,u),z),i(y,z)))).
2046 [hyper,135,1960,1217] P(j(j(x,a(x,y)),i(x,a(x,y)))).
2077 [hyper,135,730,1974] P(j(j(i(j(x,y),y),z),j(x,z))).
2212 [hyper,134,2021,503] P(i(x,j(y,a(y,x)))).
2225 [hyper,134,2021,137] P(i(i(x,i(y,z)),j(i(x,y),i(x,z)))).
2326 [hyper,135,2077,1566] P(j(x,i(j(x,y),i(z,y)))).
2329 [hyper,135,2077,1282] P(j(x,j(i(y,j(x,z)),i(y,z)))).
2347 [hyper,135,1282,2212] P(j(i(x,y),i(x,j(z,a(z,y))))).
2504 [hyper,134,284,2225] P(i(i(x,i(y,i(z,u))),i(x,j(i(y,z),i(y,u))))).
2638 [hyper,135,2031,2326] P(j(j(j(x,y),i(z,u)),i(j(x,u),i(z,u)))).
2761 [hyper,135,822,2329] P(j(i(x,j(y,z)),j(y,i(x,z)))).
2884 [hyper,135,2077,2347] P(j(x,i(j(x,y),j(z,a(z,y))))).
2983 [hyper,135,2638,2046] P(i(j(x,a(x,y)),i(x,a(x,y)))).
3015 [hyper,135,2761,2212] P(j(x,i(y,a(x,y)))).
3017 [hyper,135,2761,296] P(j(j(x,y),i(j(y,z),j(x,z)))).
3065 [hyper,135,1919,2884] P(i(j(x,y),j(x,a(x,y)))).
3185 [hyper,135,1282,2983] P(j(i(x,j(y,a(y,z))),i(x,i(y,a(y,z))))).
3228 [hyper,135,730,3015] P(j(j(i(x,a(y,x)),z),j(y,z))).
3364 [hyper,135,860,3017] P(j(i(x,y),i(j(y,z),j(x,z)))).
3575 [hyper,135,3185,3065] P(i(j(x,y),i(x,a(x,y)))).
3621 [hyper,135,3228,967] P(j(x,j(j(i(a(x,y),y),y),a(x,y)))).
3664 [hyper,135,3364,668] P(i(j(i(a(x,y),z),u),j(i(x,z),u))).

3741 [hyper,134,284,3575] P(i(i(x,j(y,z)),i(x,i(y,a(y,z))))).
3816 [hyper,135,485,3621] P(j(x,j(j(i(a(x,y),y),y),y))).
3930 [hyper,134,2504,3741] P(i(i(x,j(y,z)),j(i(x,y),i(x,a(y,z))))).
3997 [hyper,135,1924,3816] P(j(x,i(a(x,y),y))).
4160 [hyper,135,1020,3930] P(j(j(x,i(y,j(z,u))),j(x,j(i(y,z),i(y,a(z,u)))))).
4214 [hyper,135,730,3997] P(j(j(i(a(x,y),y),z),j(x,z))).
4465 [hyper,135,4214,301] P(j(x,i(a(x,y),j(z,y)))).
4643 [hyper,135,1919,4465] P(i(a(x,y),j(x,y))).
4803 [hyper,135,639,4643] P(i(i(j(x,y),z),i(a(x,y),z))).
4853 [hyper,135,1020,4803] P(j(j(x,i(j(y,z),u)),j(x,i(a(y,z),u)))).
4907 [hyper,135,4853,3364] P(j(i(x,y),i(a(y,z),j(x,z)))).
4935 [hyper,135,4160,4907] P(j(i(x,y),j(i(a(y,z),x),i(a(y,z),a(x,z))))).
5015 [hyper,135,822,4935] P(j(i(a(x,y),z),j(i(z,x),i(a(x,y),a(z,y))))).
5097 [hyper,134,3664,5015] P(j(i(x,y),j(i(y,x),i(a(x,z),a(y,z))))).

A 94-Step Proof in SBPC of the Fourth Theorem

----- Otter 3.3g-work, Jan 2005 -----
The process was started by wos on crush.mcs.anl.gov,
Fri Jun 5 21:11:13 2009
The command was "otter". The process ID is 23121.

----> UNIT CONFLICT at 48788.89 sec ----> 1865903 [binary,1865902.1,38.1] $ANS(4).

Length of proof is 94. Level of proof is 36.

---------------- PROOF ----------------

16 [] -P(i(x,y))| -P(x)|P(y).
17 [] -P(j(x,y))| -P(x)|P(y).
18 [] P(i(x,i(y,x))).
19 [] P(i(i(x,i(y,z)),i(i(x,y),i(x,z)))).
20 [] P(i(x,j(y,x))).
21 [] P(i(j(x,j(y,z)),j(j(x,y),j(x,z)))).
22 [] P(i(j(i(x,y),y),j(i(y,x),x))).
23 [] P(j(i(x,y),j(x,y))).
24 [] P(j(x,o(x,y))).
25 [] P(i(y,o(x,y))).
26 [] P(j(j(x,z),j(j(y,z),j(o(x,y),z)))).
27 [] P(i(a(x,y),x)).
28 [] P(j(a(x,y),y)).
29 [] P(i(i(x,y),i(i(x,z),i(x,a(y,z))))).
38 [] -P(j(i(A,B),i(a(C,A),a(C,B))))|$ANS(4).
40 [hyper,16,18,18] P(i(x,i(y,i(z,y)))).
41 [hyper,16,19,19] P(i(i(i(x,i(y,z)),i(x,y)),i(i(x,i(y,z)),i(x,z)))).
47 [hyper,17,23,22] P(j(j(i(x,y),y),j(i(y,x),x))).
48 [hyper,17,23,21] P(j(j(x,j(y,z)),j(j(x,y),j(x,z)))).
49 [hyper,17,23,20] P(j(x,j(y,x))).
51 [hyper,17,23,18] P(j(x,i(y,x))).
91 [hyper,16,19,40] P(i(i(x,y),i(x,i(z,y)))).
93 [hyper,16,41,40] P(i(i(x,i(i(y,x),z)),i(x,z))).
109 [hyper,17,49,49] P(j(x,j(y,j(z,y)))).

131 [hyper,17,23,91] P(j(i(x,y),i(x,i(z,y)))).
141 [hyper,16,91,19] P(i(i(x,i(y,z)),i(u,i(i(x,y),i(x,z))))).
150 [hyper,16,18,93] P(i(x,i(i(y,i(i(z,y),u)),i(y,u)))).
152 [hyper,16,93,18] P(i(x,x)).
207 [hyper,16,93,150] P(i(i(i(x,y),z),i(y,z))).
208 [hyper,16,19,150] P(i(i(x,i(y,i(i(z,y),u))),i(x,i(y,u)))).
220 [hyper,17,23,152] P(j(x,x)).
223 [hyper,16,29,152] P(i(i(x,y),i(x,a(x,y)))).
243 [hyper,16,207,19] P(i(i(x,y),i(i(z,x),i(z,y)))).
252 [hyper,16,208,141] P(i(i(x,i(y,z)),i(y,i(x,z)))).
257 [hyper,17,26,220] P(j(j(x,y),j(o(y,x),y))).
288 [hyper,16,207,223] P(i(x,i(y,a(y,x)))).
318 [hyper,16,243,243] P(i(i(x,i(y,z)),i(x,i(i(u,y),i(u,z))))).
328 [hyper,16,243,21] P(i(i(x,j(y,j(z,u))),i(x,j(j(y,z),j(y,u))))).
329 [hyper,16,243,20] P(i(i(x,y),i(x,j(z,y)))).
348 [hyper,16,252,243] P(i(i(x,y),i(i(y,z),i(x,z)))).
455 [hyper,16,252,288] P(i(x,i(y,a(x,y)))).
520 [hyper,16,328,20] P(i(j(x,y),j(j(z,x),j(z,y)))).
538 [hyper,17,23,329] P(j(i(x,y),i(x,j(z,y)))).
572 [hyper,17,23,348] P(j(i(x,y),i(i(y,z),i(x,z)))).
574 [hyper,16,318,348] P(i(i(x,y),i(i(z,i(y,u)),i(z,i(x,u))))).
581 [hyper,16,348,329] P(i(i(i(x,j(y,z)),u),i(i(x,z),u))).
713 [hyper,17,23,455] P(j(x,i(y,a(x,y)))).
807 [hyper,17,23,520] P(j(j(x,y),j(j(z,x),j(z,y)))).
818 [hyper,16,520,51] P(j(j(x,y),j(x,i(z,y)))).
819 [hyper,16,520,28] P(j(j(x,a(y,z)),j(x,z))).
825 [hyper,16,520,538] P(j(j(x,i(y,z)),j(x,i(y,j(u,z))))).
1099 [hyper,16,21,807] P(j(j(j(x,y),j(z,x)),j(j(x,y),j(z,y)))).
1189 [hyper,16,520,819] P(j(j(x,j(y,a(z,u))),j(x,j(y,u)))).
1215 [hyper,17,825,713] P(j(x,i(y,j(z,a(x,y))))).
1452 [hyper,17,1099,109] P(j(j(j(x,y),z),j(y,z))).
1903 [hyper,17,807,1452] P(j(j(x,j(j(y,z),u)),j(x,j(z,u)))).
1916 [hyper,17,1452,1099] P(j(j(x,y),j(j(y,z),j(x,z)))).
1926 [hyper,17,1452,47] P(j(x,j(i(x,y),y))).
2054 [hyper,17,1903,48] P(j(j(x,j(y,z)),j(y,j(x,z)))).
2057 [hyper,17,1916,1916] P(j(j(j(j(x,y),j(z,y)),u),j(j(z,x),u))).
2221 [hyper,17,807,1926] P(j(j(x,y),j(x,j(i(y,z),z)))).
2539 [hyper,17,2054,2054] P(j(x,j(j(y,j(x,z)),j(y,z)))).
2569 [hyper,17,2054,47] P(j(i(x,y),j(j(i(y,x),x),y))).
2694 [hyper,17,2221,24] P(j(x,j(i(o(x,y),z),z))).
2769 [hyper,17,48,2539] P(j(j(x,j(y,j(x,z))),j(x,j(y,z)))).
2816 [hyper,17,807,2569] P(j(j(x,i(y,z)),j(x,j(j(i(z,y),y),z)))).
2821 [hyper,17,2569,27] P(j(j(i(x,a(x,y)),a(x,y)),x)).
2822 [hyper,17,2569,25] P(j(j(i(o(x,y),y),y),o(x,y))).
2823 [hyper,17,2569,20] P(j(j(i(j(x,y),y),y),j(x,y))).
3031 [hyper,17,2769,2694] P(j(x,j(i(o(x,y),j(x,z)),z))).
3070 [hyper,17,2816,713] P(j(x,j(j(i(a(x,y),y),y),a(x,y)))).
3141 [hyper,17,2054,2823] P(j(x,j(j(i(j(x,y),y),y),y))).
3195 [hyper,17,2054,3031] P(j(i(o(x,y),j(x,z)),j(x,z))).
3286 [hyper,17,1189,3070] P(j(x,j(j(i(a(x,y),y),y),y))).
3354 [hyper,17,2822,3195] P(o(x,j(x,y))).
3496 [hyper,17,2539,3354] P(j(j(x,j(o(y,j(y,z)),u)),j(x,u))).
3547 [hyper,17,3496,257] P(j(j(j(x,y),x),x)).

3571 [hyper,17,807,3547] P(j(j(x,j(j(y,z),y)),j(x,y))).
3637 [hyper,17,3571,818] P(j(j(j(i(x,y),z),y),i(x,y))).
3720 [hyper,17,2057,3637] P(j(j(x,i(y,j(x,z))),i(y,j(x,z)))).
3724 [hyper,17,807,3637] P(j(j(x,j(j(i(y,z),u),z)),j(x,i(y,z)))).
3781 [hyper,17,3720,1215] P(i(x,j(y,a(y,x)))).
3783 [hyper,17,2057,3724] P(j(j(j(i(x,y),z),u),j(j(u,y),i(x,y)))).
3784 [hyper,17,3724,3286] P(j(x,i(a(x,y),y))).
3785 [hyper,17,3724,3141] P(j(x,i(j(x,y),y))).
4164 [hyper,17,2057,3783] P(j(j(x,i(y,z)),j(j(j(x,u),z),i(y,z)))).
4169 [hyper,17,3783,2821] P(j(j(x,a(x,y)),i(x,a(x,y)))).
4212 [hyper,17,825,3784] P(j(x,i(a(x,y),j(z,y)))).
4383 [hyper,17,1916,3785] P(j(j(i(j(x,y),y),z),j(x,z))).
4583 [hyper,17,4164,572] P(j(j(j(i(x,y),z),i(x,u)),i(i(y,u),i(x,u)))).
4634 [hyper,17,3720,4212] P(i(a(x,y),j(x,y))).
4751 [hyper,17,818,4383] P(j(j(i(j(x,y),y),z),i(u,j(x,z)))).
4772 [hyper,17,4383,131] P(j(x,i(j(x,y),i(z,y)))).
4845 [hyper,16,574,4634] P(i(i(x,i(j(y,z),u)),i(x,i(a(y,z),u)))).
4895 [hyper,17,4583,4751] P(i(i(x,j(y,z)),i(j(y,x),j(y,z)))).
4919 [hyper,17,4164,4772] P(j(j(j(x,y),i(z,u)),i(j(x,u),i(z,u)))).
5273 [hyper,16,581,4895] P(i(i(x,y),i(j(z,x),j(z,y)))).
5277 [hyper,16,4895,3781] P(i(j(x,y),j(x,a(x,y)))).
5485 [hyper,16,4845,5273] P(i(i(x,y),i(a(z,x),j(z,y)))).
179728 [hyper,17,4919,4169] P(i(j(x,a(x,y)),i(x,a(x,y)))).
179753 [hyper,16,243,179728] P(i(i(x,j(y,a(y,z))),i(x,i(y,a(y,z))))).
179759 [hyper,16,179753,5277] P(i(j(x,y),i(x,a(x,y)))).
179800 [hyper,16,252,179759] P(i(x,i(j(x,y),a(x,y)))).
180025 [hyper,16,243,179800] P(i(i(x,y),i(x,i(j(y,z),a(y,z))))).
1865702 [hyper,16,180025,27] P(i(a(x,y),i(j(x,z),a(x,z)))).
1865720 [hyper,16,19,1865702] P(i(i(a(x,y),j(x,z)),i(a(x,y),a(x,z)))).
1865722 [hyper,16,243,1865720] P(i(i(x,i(a(y,z),j(y,u))),i(x,i(a(y,z),a(y,u))))).
1865892 [hyper,16,1865722,5485] P(i(i(x,y),i(a(z,x),a(z,y)))).
1865902 [hyper,17,23,1865892] P(j(i(x,y),i(a(z,x),a(z,y)))).

In many ways, a breadth-first (level-saturation) approach is appealing, and various effective programs rely on this type of search. Nevertheless, the levels cited for the four proofs of the cited theorems in *SBPC suggest that the proof of any of the four might have been out of reach of this type of search*.

7. Overview

In this notebook, you can, and perhaps wth pleasure and surprise, read about the discovery of new theorems, where the impetus did not come directly from a person. Indeed, what is presented here is in the spirit of theorem finding. The proofs that are occasionally offered by a program, such as McCune's automated reasoning program, sometimes contain treasure different from the goal that prompted the corresponding study. Sometimes the output file contains valuable and hidden information. In that regard, a long run ordinarily yields a large output file. An expert might, when browsing in the output, find lemmas and even theorems that are unexpected.

The studies on which this notebook is based resulted directly from M. Spinks and his request for shorter proofs than he had in hand. R. Veroff, with his powerful sketches approach, had produced many proofs, some that would have been quite difficult to complete. I relied on many of Veroff's proofs throughout my research. Although I was primarily seeking shorter proofs than those in hand, as you learn here, I found real treasure, treasure in the form of unexpected dependencies in various areas of logic. I think it safe to say, in the spirit of accuracy, that OTTER (McCune's program) found much of what is offered here. I cannot help but wonder about other dependencies, say, in *SBPC* when the focus is on twelve axioms rather than on fifteen. Perhaps one of you can supply appropriate models establishing independence or appropriate proofs establishing dependencies. Of course, I have in mind the omission of each of *A*3, *A*6, and *A*7. Each of these three cited axioms is dependent on the remaining twelve. Further, the proof of the dependence of *A*7 in the *SBPC* logic does not require the use (at the axiomatic level) of any of the three axioms in the function *a* for logical **and**. From what I know, before OTTER and I entered the game, the dependencies cited here and discovered some years ago were unknown to the field.